Solution to Problem 1378
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Triangular Numbers

1378. *Proposed by Stephen G. Penrice, Arizona State University, Tempe, Arizona*

Let $(i)_j$ denote the falling product $i \cdot (i-1) \cdot \ldots \cdot (i-j+1)$ and let $(i)_0 = 1$. Show that for all positive integers n and k

$$\frac{(n+k)_{k+1}}{2(k!)^2} \sum_{i=1}^{k} (k)_{i-1}(n+k-i)_{k-i}$$

is a triangular number.

SOLUTION

Writing $(i)_j$ as $i!/(i-j)!$, we may write the given expression as

$$\frac{1}{2}\binom{n+k}{n} \sum_{i=1}^{k} \binom{n+k-i}{n-1}$$

Now using the standard additive property of binomial coefficients, we can write this as

$$\frac{1}{2}\binom{n+k}{n} \sum_{i=1}^{k} \left(\binom{n+k-i+1}{n} - \binom{n+k-i}{n}\right)$$

which telescopes to

$$\frac{1}{2}\binom{n+k}{n} \left(\binom{n+k}{n} - 1\right),$$

a triangular number.