Solution to Problem 1529
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An Integral Sum of Cube Roots

1529. *Proposed by David C. Kay, University of North Carolina at Asheville, North Carolina.*

For what positive numbers is

$$\sqrt[3]{2+\sqrt{a}}+\sqrt[3]{2-\sqrt{a}}$$

an integer?

SOLUTION: Let $P=\sqrt[3]{2+\sqrt{a}}$, $Q=\sqrt[3]{2-\sqrt{a}}$, and $x=P+Q$. Since $|2+\sqrt{a}|>|2-\sqrt{a}|$ then $x=P+Q>0$.

Cubing we get

$$\begin{aligned} x^3 &= P^3+3P^2Q+3PQ^2+Q^3 \\ x^3 &= 4+3PQ(P+Q) \\ x^3 &= 4+3x\cdot\sqrt[3]{4-a}. \end{aligned}$$

Solving for a yields

$$a=4-\left(\frac{x^2}{3}-\frac{4}{3x}\right)^3 \tag{1}$$

and

$$\frac{da}{dx}=-3\left(\frac{x^2}{3}-\frac{4}{3x}\right)^2\left(\frac{2x}{3}+\frac{4}{3x^2}\right)<0.$$

Since we want x to be a positive integer, we can plug into (1) and get

$$x = 1 \Rightarrow a = 5$$
$$x = 2 \Rightarrow a = 100/27$$
$$x = 3 \Rightarrow a < 0.$$

Thus, there are two values of a which satisfy the problem. They are $a = 5$ and $a = 100/27$.