Solution to Problem 1533
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**A Quadratic Recurrence Relation**

**1533.** *Proposed by Joaquín Gómez Rey, I. B. "Luis Buñuel," Alcorcón, Madrid, Spain.*

Solve the recurrence relation

$$a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k}$$

in terms of $a_0$.

SOLUTION: An examination of the first few terms leads to the conjecture that

$$a_n = n! \cdot a_0^{n+1}$$

The proof will proceed by induction. Assume that

$$\begin{aligned} a_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} = (n+1)! \cdot a_0^{n+2} \\ a_{n+2} &= \sum_{k=0}^{n+1} \binom{n+1}{k} a_k a_{n+1-k} \end{aligned}$$

Using the induction hypothesis,

$$\begin{aligned}
a_{n+2} &= \sum_{k=0}^{n+1} \binom{n+1}{k} k! a_0^{k+1} \cdot (n+1-k)! a_0^{n+2-k} \\
&= \sum_{k=0}^{n+1} \frac{(n+1)!}{k!(n+1-k)!} k!(n+1-k)! a_0^{n+3} \\
&= a_0^{n+3} \sum_{k=0}^{n+1} (n+1)! = (n+2)! \cdot a_0^{n+3}.
\end{aligned}$$