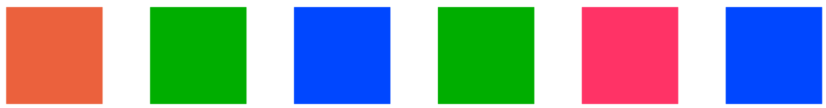
Self-Descriptive Sequences

The Problem

In 1958, a mathematician named C. D. Langford noticed his child playing with colored blocks. By chance the child had arranged the blocks into a particularly interesting sequence.



Langford translated the problem into numbers. Using the pattern of blocks shown above, rearrange the sequence $1, 1, 2, 2, 3, 3$ so that there is one number between the two 1's, two numbers between the two 2's, and three numbers between the two 3's.

Suggested Materials

Colored blocks to illustrate the initial problem would be nice for younger grades. Pencil and paper are sufficient for older kids.

Prerequisites

It is not necessary that the students know that a sequence is a function whose domain is the set of natural numbers. However students should understand that a sequence is a string of numbers (separated by commas) that follow a well-defined pattern.

Guided Exploration

Students might be broken into groups of two's and three's and asked to precisely state and solve the problem for the cases $1, 1, 2, 2, 3, 3, 4, 4$ and $1, 1, 2, 2, 3, 3, 4, 4, 5, 5$.

Concluding the Exploration

After spending five to ten minutes introducing the problem, the teacher can let the students work in their groups for ten to fifteen minutes attepting to solve the cases in the guided exploration. Ask the students to observe any patterns they notice and to describe what difficulties they encounter. Don't let the students work too long on the case $n = 5$ as they may get frustrated.

A Mathematical Approach

The sequences that we have studied in this unit are called $(2, n)$ Langford sequences. In general we have the following:

DEFINITION: A (k, n) Langford sequence is a rearrangement of the sequence

$$1, 1, \ldots, 1, 2, 2, \ldots, 2, 3, 3, \ldots, 3, \ldots, n, n, \ldots, n$$
$$(k \text{ copies of each integer from 1 to } n)$$

such that between each occurrence of the integer i there are i integers.

We can characterize those values of n such that a $(2, n)$ Langford sequence exists. Let a_1 indicate the *position* of the first 1. Then $a_1 + 2$ must indicate the position of the second 1 since they must be separated by one integer. In general, the two appearances of the integer i will occur at *positions* a_i and $a_i + (i+1)$. In a $(2, n)$ Langford sequence there are $2n$ positions to be filled. Thus we have

Number	1	2	3	...	n
1^{st} appearance	a_1	a_2	a_3	...	a_n
2^{nd} appearance	$a_1 + 2$	$a_2 + 3$	$a_3 + 4$	...	$a_n + (n+1)$

Adding all these positions we get

$$2a_1 + 2a_2 + \ldots + 2a_n + (2 + 3 + \ldots + (n+1)) = 1 + 2 + \ldots + (2n)$$

or on collecting terms

$$\begin{aligned} 2(a_1 + a_2 + \ldots + a_n) + \frac{(n+1)(n+2)}{2} - 1 &= \frac{(2n)(2n+1)}{2} \\ a_1 + a_2 + \ldots + a_n &= \frac{n(3n-1)}{4} \end{aligned}$$

Notice that n and $3n-1$ have opposite parity. Since the left hand side is a sum of integers, then we must have that $n(3n-1) \equiv 0 \pmod 4$. The least positive solutons to this modular equation are $n \equiv 0$ and $n \equiv 3 \equiv -1 \pmod 4$. Thus, $n = 4k$ or $n = 4k+3 \equiv 4k-1 \pmod 4$. Since we know that solutions exist for $n = 3$ and $n = 4$, we choose as our solutions $n = 4k$ and $n = 4k-1$ so that the parameter k can start at the same value $k = 1$.

Further Problems for Review or Assessment

You might wish to have enterprising students see if they can find an example of a $(3, n)$ or $(4, n)$ Langford sequence.

Additional Information

Examples of Langford sequences

$(2,3)$	$\Rightarrow$	3,1,2,1,3,2
$(2,4)$	$\Rightarrow$	4,1,3,1,2,4,3,2
$(2,7)$	$\Rightarrow$	7,3,6,2,5,3,2,4,7,6,5,1,4,1
$(2,8)$	$\Rightarrow$	8,3,7,2,6,3,2,4,5,8,7,6,4,1,5,1
$(3,9)$	$\Rightarrow$	1,9,1,6,1,8,2,5,7,2,6,9,2,5,8,4,7,6,3,5,4,9,3,8,7,4,3
$(4,24)$	$\Rightarrow$	8,17,1,3,1,16,1,3,1,8,10,3,23,21,19,3,11,24,8,17,2,10,16,2, 20,22,2,8,11,2,13,18,10,15,19,21,23,17,12,16,11,5,24,10,13,20,14,5, 22,15,18,12,11,5,19,17,16,23,13,5,23,14,7,9,12,15,20,24,6,18,7,22, 13,9,19,6,14,12,7,21,4,15,6,9,23,4,7,20,18,6,4,14,24,9,22,4

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