Transformations

Let $\left(\frac{dz}{dx}\right)^n = \sum_{k=1}^{K} g_k(x)\cdot z^{m_k}$. Then $z^\alpha \cdot \left(\frac{dz}{dx}\right)^n = \sum_{k=1}^{K} g_k(x)\cdot z^{m_k+\alpha}$. So $\left(z^{\alpha/n}\cdot \frac{dz}{dx}\right)^n = \sum_{k=1}^{K} g_k(x)\cdot z^{m_k+\alpha}$. Let $u \equiv z^{\alpha/n+1} = z^{\frac{\alpha+n}{n}}$. So $z = u^{\frac{n}{\alpha+n}}$ and $\frac{du}{dx} = \frac{\alpha+n}{n}\cdot z^{\frac{\alpha}{n}}\cdot\frac{dz}{dx}$. So $\left(\frac{du}{dx}\right)^n = \sum_{k=1}^{K}\left(\frac{\alpha+n}{n}\right)^n g_k(x)\cdot z^{m_k+\alpha}$. So $\left(\frac{du}{dx}\right)^n = \sum_{k=1}^{K}\left(\frac{\alpha+n}{n}\right)^n g_k(x)\cdot u^{\frac{n\cdot(\alpha+m_k)}{\alpha+n}}$.

Let $z = F(g_1(x),\ldots,g_K(x),m_1,\ldots,m_K)$. Then

$$u = F(\left(\frac{\alpha+n}{n}\right)^n\cdot g_1(x),\ldots,\left(\frac{\alpha+n}{n}\right)^n\cdot g_K(x),\frac{n\cdot(m_1+\alpha)}{n+\alpha},\ldots,\frac{n\cdot(m_K+\alpha)}{n+\alpha}).$$

So F satisfies the functional, *non-differential* relation

$$\left(F(\left(\frac{\alpha+n}{n}\right)^n\cdot g_1(x),\ldots,\left(\frac{\alpha+n}{n}\right)^n\cdot g_K(x),\frac{n\cdot(m_1+\alpha)}{n+\alpha},\ldots,\frac{n\cdot(m_K+\alpha)}{n+\alpha})\right)^{\frac{n}{\alpha+n}}$$
$$= F(g_1(x),\ldots,g_K(x),m_1,\ldots,m_K)$$

Observe that one may permute the functions and corresponding powers. Note also that if $z = F(g_1(x),\ldots,g_K(x),m_1,\ldots,m_K)$ satisfies this nonlinear functional equation, then any power of it, $H(g_1(x),\ldots,g_K(x),m_1,\ldots,m_K)\equiv(F(g_1(x),\ldots,g_K(x),m_1,\ldots,m_K))^\beta$, satisfies the same nonlinear functional equation, because

$$\left(H(\left(\frac{\alpha+n}{n}\right)^n\cdot g_1(x),\ldots,\left(\frac{\alpha+n}{n}\right)^n\cdot g_K(x),\frac{n\cdot(m_1+\alpha)}{n+\alpha},\ldots,\frac{n\cdot(m_K+\alpha)}{n+\alpha})\right)^{\frac{n}{\alpha+n}}$$
$$\left(F(\left(\frac{\alpha+n}{n}\right)^n\cdot g_1(x),\ldots,\left(\frac{\alpha+n}{n}\right)^n\cdot g_K(x),\frac{n\cdot(m_1+\alpha)}{n+\alpha},\ldots,\frac{n\cdot(m_K+\alpha)}{n+\alpha})\right)^{\beta\cdot\frac{n}{\alpha+n}}$$
$$= (F(g_1(x),\ldots,g_K(x),m_1,\ldots,m_K))^\beta$$
$$= H(g_1(x),\ldots,g_K(x),m_1,\ldots,m_K)$$

The Bernoulli Equation

Set $K=2$, $m_1=1$, $n=1$, $m_2=m$. Then $\frac{dz}{dx} = g_1(x)\cdot z + g_2(x)\cdot z^m$

$\left(F((\alpha+1)\cdot g_1(x),(\alpha+1)\cdot g_2(x),1,\frac{\alpha+m}{\alpha+1})\right)^{\frac{1}{\alpha+1}} = F(g_1(x),g_2(x),1,m)$. We may simplify F by expressing F as a function of 3 variables, not 4. Thus, $z = F(g_1(x),g_2(x),m)$. Then for almost any α , $\left(F((\alpha+1)\cdot g_1(x),(\alpha+1)\cdot g_2(x),\frac{\alpha+m}{\alpha+1})\right)^{\frac{1}{\alpha+1}} = F(g_1(x),g_2(x),m)$ .The substitution $z = u^{\frac{1}{1-m}}$ leads to the linear equation $\frac{du}{dx} = (1-m)\cdot g_1(x)\cdot u + (1-m)\cdot g_2(x)$. Thus for almost any α,

$$\left(F((\alpha+1)\cdot g_1(x),(\alpha+1)\cdot g_2(x),\frac{\alpha+m}{\alpha+1}) \right)^{\frac{1-m}{1+\alpha}} = (F(g_1(x),g_2(x),m))^{1-m} = u(g_1(x),g_2(x),m).$$

Choose $\alpha = -m$. We get $F((1-m)\cdot g_1(x),(1-m)\cdot g_2(x),0) = u(g_1(x),g_2(x),m)$. But $(F(g_1(x),g_2(x),m))^{1-m} = u(g_1(x),g_2(x),m)$ implies, by setting $m = 0$, that $(F(g_1(x),g_2(x),0))^1 = u(g_1(x),g_2(x),0)$. Hence $F((1-m)\cdot g_1(x),(1-m)\cdot g_2(x),0) = u((1-m)\cdot g_1(x),(1-m)\cdot g_2(x),0)$. Hence $u(g_1(x),g_2(x),m) = u((1-m)\cdot g_1(x),(1-m)\cdot g_2(x),0)$. Clearly, this functional relation is satisfied by the linear differential equation $\frac{du}{dx} = (1-m)\cdot g_1(x)\cdot u + (1-m)\cdot g_2(x)$.

The relation $u(g_1(x),g_2(x),m) = u((1-m)\cdot g_1(x),(1-m)\cdot g_2(x),0)$ is the same as $u(s,t,m) = u((1-m)\cdot s,(1-m)\cdot t,0)$ for any s,t. So

$$F(g_1,g_2,m) = \left[z_0 + (1-m)\int_{t=0}^{t=x} g_2(t)\cdot \exp\left((m-1)\int_{\theta=0}^{\theta=t} g_1(\theta)d\theta \right) dt \right]^{1/(1-m)} \cdot \exp\left(\int_{\theta=0}^{\theta=x} g_1(\theta)d\theta \right)$$

The Abel Equation

Set $n=1$, $K=2$, $g_1(x)=u$, $g_2(x)=v$, $m_1=m$, $m_2=n$ in $\left(\frac{dz}{dx}\right)^n = \sum_{k=1}^{K} g_k(x)\cdot z^{m_k}$ to get $\frac{dz}{dx} = u\cdot z^m + v\cdot z^n$. Then $\left(F((1+\alpha)\cdot u,(1+\alpha)\cdot v,\frac{m+\alpha}{1+\alpha},\frac{n+\alpha}{1+\alpha}) \right)^{\frac{1}{1+\alpha}} = F(u,v,m,n)$. So $F(\alpha\cdot u,\alpha\cdot v,\frac{m-1}{\alpha}+1,\frac{n-1}{\alpha}+1) = (F(u,v,m,n))^{\alpha}$. We might face one problem: when we write this functional equation in terms of u and v as if they are independent variables, implying that u is constant with respect to v and vice versa, then we might get one general solution $F(u,v,m,n)$ for which substituting $g_1(x)$ for u and $g_2(x)$ for v will certainly satisfy the functional relation above. However, there may exist other solutions, $S(g_1(x),g_2(x),m,n)$, which are *not* specializations of $F(u,v,m,n)$ and which work *only* with $g_1(x)$ substituted for u and $g_2(x)$ for v.

Since $F(u,v,m,n) = \left(F(\alpha\cdot u,\alpha\cdot v,\frac{m-1}{\alpha}+1,\frac{n-1}{\alpha}+1) \right)^{1/\alpha}$, then $\ln(F(u,v,m,n)) = \frac{1}{\alpha}\ln\left(F(\alpha\cdot u,\alpha\cdot v,\frac{m-1}{\alpha}+1,\frac{n-1}{\alpha}+1) \right)$. Define $a \equiv \alpha\cdot u$, $b \equiv \alpha\cdot v$, $c \equiv \frac{m-1}{\alpha}+1$, and $d \equiv \frac{n-1}{\alpha}+1$. Now, differentiate this equation with respect to each of the variables u,v,m,n in turn. We get 4 equations.

$$\frac{F_u(u,v,m,n)}{F(u,v,m,n)}=\frac{1}{\alpha}\cdot\frac{F_a(a,b,c,d)}{F(a,b,c,d)}\cdot\frac{\partial a}{\partial u}=\frac{F_a(a,b,c,d)}{F(a,b,c,d)},\ \frac{F_v(u,v,m,n)}{F(u,v,m,n)}=\frac{F_b(a,b,c,d)}{F(a,b,c,d)},$$

$$\frac{F_m(u,v,m,n)}{F(u,v,m,n)}=\frac{1}{\alpha}\cdot\frac{F_c(a,b,c,d)}{F(a,b,c,d)}\cdot\frac{\partial c}{\partial m}=\frac{1}{\alpha^2}\frac{F_c(a,b,c,d)}{F(a,b,c,d)},\ \frac{F_n(u,v,m,n)}{F(u,v,m,n)}=\frac{1}{\alpha^2}\cdot\frac{F_d(a,b,c,d)}{F(a,b,c,d)}.$$

If we define $G(u,v,m,n)\equiv\dfrac{F_u(u,v,m,n)}{F(u,v,m,n)}=\dfrac{F_a(a,b,c,d)}{F(a,b,c,d)}$, observe that $G(u,v,m,n)$ is invariant under the *simultaenous* transformations $u\leftrightarrow a$, $v\leftrightarrow b$, $m\leftrightarrow c$, and $n\leftrightarrow d$ including in the subscript denoting partial derivative. Similarly, we may define

$$H(u,v,m,n)\equiv\frac{F_v(u,v,m,n)}{F(u,v,m,n)}=\frac{F_b(a,b,c,d)}{F(a,b,c,d)}=H(a,b,c,d),$$

$$I(u,v,m,n)\equiv\frac{F_m(u,v,m,n)}{F(u,v,m,n)}=\frac{1}{\alpha^2}\frac{F_c(a,b,c,d)}{F(a,b,c,d)}=\frac{1}{\alpha^2}\cdot I(a,b,c,d),$$

$J(u,v,m,n)\equiv\dfrac{F_n(u,v,m,n)}{F(u,v,m,n)}=\dfrac{1}{\alpha^2}\dfrac{F_d(a,b,c,d)}{F(a,b,c,d)}=\dfrac{1}{\alpha^2}\cdot J(a,b,c,d)$, such that H is invariant under these 4 simultaneous transformations, while I and J each get multiplied by a factor of α^{-2}.

We seek the most general forms of these four functions as possible. Take total derivatives of $G(u,v,m,n)=G(a,b,c,d)=G(\alpha\cdot u,\alpha\cdot v,\frac{m-1}{\alpha}+1,\frac{n-1}{\alpha}+1)$ with respect to α. We get $0=G_a(a,b,c,d)\cdot u+G_b(a,b,c,d)\cdot v-G_c(a,b,c,d)\cdot\frac{m-1}{\alpha^2}-G_d(a,b,c,d)\cdot\frac{n-1}{\alpha^2}$.

Differentiate as many times with respect to α as we need to, then set α to 1. Question: will subsequent differentiations with respect to α yield *independent* PDEs?

If we set α to 1 in this PDE we get

$$0=G_u(u,v,m,n)\cdot u+G_v(u,v,m,n)\cdot v-G_m(u,v,m,n)\cdot(m-1)-G_n(u,v,m,n)\cdot(n-1).$$

Similarly, we will get

$$0=H_u(u,v,m,n)\cdot u+H_v(u,v,m,n)\cdot v-H_m(u,v,m,n)\cdot(m-1)-H_n(u,v,m,n)\cdot(n-1).$$

The general solution is $G(u,v,m,n)=\sum\limits_{\substack{i,j,k,l\\ \ni i+j-k-l=0}}{}_G c_{i,j,k,l}\cdot u^i\cdot v^j\cdot(m-1)^k\cdot(n-1)^l$ and

$$H(u,v,m,n)=\sum_{\substack{i,j,k,l\\ \ni i+j-k-l=0}}{}_H c_{i,j,k,l}\cdot u^i\cdot v^j\cdot(m-1)^k\cdot(n-1)^l.$$

A second differentiation with respect to α yields

$$0=G_{aa}\cdot u^2+G_{bb}\cdot v^2+G_{cc}\cdot(m-1)^2\alpha^{-4}+G_{dd}\cdot(n-1)^2\alpha^{-4}$$
$$+2\cdot\begin{bmatrix}G_{ab}\cdot uv+G_{cd}\cdot(m-1)(n-1)\alpha^{-4} \\ -\alpha^{-2}\cdot\{G_{ac}\cdot u(m-1)+G_{ad}\cdot u(n-1)+G_{bc}\cdot v(m-1)+G_{bd}\cdot v(n-1)\end{bmatrix}$$
$$+2\cdot\alpha^{-3}[G_c\cdot(m-1)+G_d\cdot(n-1)]$$

Set set α to 1 to get

$$\begin{aligned} 0 &= G_{uu}\cdot u^2 + G_{vv}\cdot v^2 + G_{mm}\cdot(m-1)^2 + G_{nn}\cdot(n-1)^2 \\ &+2\cdot\begin{bmatrix} G_{uv}\cdot uv + G_{mn}\cdot(m-1)(n-1) \\ -\{G_{um}\cdot u(m-1) + G_{un}\cdot u(n-1) + G_{vm}\cdot v(m-1) + G_{vn}\cdot v(n-1) \end{bmatrix} \\ &+2\cdot[G_m\cdot(m-1) + G_n\cdot(n-1)] \end{aligned}$$

Reading off the powers, we can guess that any function $G(u,v,m,n) = \sum_{i,j,k,l} {}_G c_{i,j,k,l}\cdot u^i\cdot v^j\cdot(m-1)^k\cdot(n-1)^l$ whose coefficients ${}_G c_{i,j,k,l}$ are all 0 unless $\begin{aligned} &i\cdot(i-1) + j\cdot(j-1) + k\cdot(k-1) + l\cdot(l-1) \\ &+2\cdot(i\cdot j + k\cdot l - i\cdot k - i\cdot l - j\cdot k - j\cdot l) + 2\cdot(k+l) = 0 \end{aligned}$ which simplifies to $((i+j)-(k+l))^2 + ((k+l)-(i+j))$. So, third equation is satsified whenever $i+j = k+l$ or $i+j = k+l+1$. Hence, these two PDEs are *dependent*.

In fact, one sees that *all* differential consequences of this PDE with respect to α will have solutions which are functions of $i+j$ and $k+l$ separately.

Also, $\begin{aligned} &0 = -2\alpha^{-3}\cdot J(a,b,c,d) + \alpha^{-2}\cdot\{J_a(a,b,c,d)\cdot u + J_b(a,b,c,d)\cdot v \\ &-\alpha^{-2}\cdot[J_c(a,b,c,d)\cdot(m-1) + J_d(a,b,c,d)\cdot(n-1)]\} \end{aligned}$ Set α to 1 in this PDE. We get $\begin{aligned} &0 = -2\cdot J(u,v,m,n) + \{J_u(u,v,m,n)\cdot u + J_v(u,v,m,n)\cdot v \\ &-[J_m(u,v,m,n)\cdot(m-1) + J_n(u,v,m,n)\cdot(n-1)]\} \end{aligned}$.

Before we do a lot of work finding the most general solution for $G(u,v,m,n)$, observe that we will get similar *linear partial differential equations* for $H(u,v,m,n)$, $I(u,v,m,n)$, and $J(u,v,m,n)$. These general solutions will have to be further constrained by the relations

$$\begin{aligned} &F(u,v,m,n) = F(0,v,m,n)\cdot\exp\left(\int_0^u G(\theta,v,m,n)\cdot d\theta\right) \\ &= F(u,0,m,n)\cdot\exp\left(\int_0^v H(u,\theta,m,n)\cdot d\theta\right) \\ &= F(u,v,1,n)\cdot\exp\left(\int_1^m I(u,v,\theta,n)\cdot d\theta\right) \\ &= F(u,v,m,1)\cdot\exp\left(\int_1^n J(u,v,m,\theta)\cdot d\theta\right) \end{aligned}$$

We know the solutions $F(u,v,1,n)$ and $F(u,v,m,1)$. These are the Bernoulli equations. Similarly, $F(0,v,m,n)$ and $F(u,0,m,n)$ "collapse" to solutions of easy cases of the Abel equation. The definitions of G, H, I, and J relate them in the following 6 ways: $G_v = H_u$, $G_m = I_u$, $G_n = J_u$, $H_m = I_v$, $H_n = J_v$, $I_n = J_m$.

$$\sum_{\substack{i,j,k,l \\ \ni i+j+1-k-l=0}} {}_{G}c_{i,j+1,k,l} \cdot (j+1) \cdot u^i \cdot v^j \cdot (m-1)^k \cdot (n-1)^l = G_v$$

So

$$= H_u = \sum_{\substack{i,j,k,l \\ \ni i+1+j-k-l=0}} {}_{H}c_{i+1,j,k,l} \cdot (i+1) \cdot u^i \cdot v^j \cdot (m-1)^k \cdot (n-1)^l$$

So ${}_{G}c_{i,j+1,k,l} = {}_{H}c_{i+1,j,k,l}$. Similarly ${}_{G}c_{i,j,k+1,l} = {}_{I}c_{i+1,j,k,l}$, ${}_{G}c_{i,j,k,l+1} = {}_{J}c_{i+1,j,k,l}$, ${}_{H}c_{i,j,k+1,l} = {}_{I}c_{i,j+1,k,l}$, ${}_{H}c_{i,j,k,l+1} = {}_{J}c_{i,j+1,k,l}$, ${}_{I}c_{i,j,k,l+1} = {}_{J}c_{i,j,k+1,l}$.

We must somehow associate u with m and v with n. It *might* be possible to make this association by the formula above via the solutions $F(0,v,m,n)$ and $F(u,0,m,n)$ and the Bernoulli solutions $F(u,v,1,n)$ and $F(u,v,m,1)$. We have $F(0,v,m,n)$ is the solution of $\dfrac{dz}{dx} = v \cdot z^n$. So $F(0,v,m,n) = \left((1-n) \cdot \int\limits_{t=0}^{t=x} v(t)dt + z_0^{1-n} \right)^{1/(1-n)}$ for $n \neq 1$ and $F(0,v,m,1) = z_0 \cdot \exp\left(\int\limits_{t=0}^{t=x} v(t)dt \right)$.

And, we also have the initial condition of the original Abel differential equation $\dfrac{dz}{dx} = u \cdot z^m + v \cdot z^n$ to satisfy, namely, $z\big|_{x=0} = F(u(0),v(0),m,n)$. Since

$$F(u,v,1,n) = \left[z_0 + (1-n)\int\limits_{t=0}^{t=x} v(t) \cdot \exp\left((n-1)\int\limits_{\theta=0}^{\theta=t} u(\theta)d\theta \right)dt \right]^{1/(1-n)} \cdot \exp\left(\int\limits_{\theta=0}^{\theta=x} u(\theta)d\theta \right)$$

$$F(u,v,m,1) = \left[z_0 + (1-m)\int\limits_{t=0}^{t=x} u(t) \cdot \exp\left((m-1)\int\limits_{\theta=0}^{\theta=t} v(\theta)d\theta \right)dt \right]^{1/(1-m)} \cdot \exp\left(\int\limits_{\theta=0}^{\theta=x} v(\theta)d\theta \right)$$

$\ln(z_0) = \ln(F(u(0),v(0),m,1)) + \int\limits_1^n J(u(0),v(0),m,\theta) \cdot d\theta$. But we also have

$\ln(F(u,v,m,1)) + \int\limits_1^n J(u,v,m,\theta) \cdot d\theta = \ln(F(u,v,1,n)) + \int\limits_1^m I(u,v,\theta,n) \cdot d\theta$. So

$$\frac{1}{1-m}\ln\left[z_0 + (1-m)\int\limits_{t=0}^{t=x} u(t) \cdot \exp\left((m-1)\int\limits_{\theta=0}^{\theta=t} v(\theta)d\theta \right)dt \right] + \int\limits_{\theta=0}^{\theta=x} v(\theta)d\theta + \int\limits_1^n J(u,v,m,\theta) \cdot d\theta$$

$$= \frac{1}{1-n}\ln\left[z_0 + (1-n)\int\limits_{t=0}^{t=x} v(t) \cdot \exp\left((n-1)\int\limits_{\theta=0}^{\theta=t} u(\theta)d\theta \right)dt \right] + \int\limits_{\theta=0}^{\theta=x} u(\theta)d\theta + \int\limits_1^m I(u,v,\theta,n) \cdot d\theta$$

We also have

$$\begin{aligned} F_u \cdot \frac{du}{dx} + F_v \cdot \frac{dv}{dx} &= \frac{DF}{Dx} = u \cdot F^m + v \cdot F^n \\ &= u \cdot F(m \cdot u, m \cdot v, \frac{m-1}{m}+1, \frac{n-1}{m}+1) + v \cdot F(n \cdot u, n \cdot v, \frac{m-1}{n}+1, \frac{n-1}{n}+1) \end{aligned}$$

For any meromorphic functions $u(x)$ of x, we can express $\frac{du}{dx}$ as a function of u, say $\frac{du}{dx} = \psi(u)$. Similarly, let $\frac{dv}{dx} = \phi(v)$. So

$$F_u \cdot \psi(u) + F_v \cdot \phi(v) = u \cdot F(m \cdot u, m \cdot v, 2 - \frac{1}{m}, \frac{n-1}{m} + 1) + v \cdot F(n \cdot u, n \cdot v, \frac{m-1}{n} + 1, 2 - \frac{1}{n}).$$

Write $F(u,v,m,n) = \sum_{i,j,k,l} {}_F c_{i,j,k,l} \cdot u^i v^j m^k n^l$. Then

$$F(u,v,m,n) = \psi(u) \cdot \sum_{i,j,k,l} {}_F c_{i,j,k,l} \cdot i \cdot u^{i-1} v^j m^k n^l + \phi(v) \cdot \sum_{i,j,k,l} {}_F c_{i,j,k,l} \cdot j \cdot u^i v^{j-1} m^k n^l$$

$$+ \sum_{i,j,k,l} {}_F c_{i,j,k,l} \cdot u^{i+1} v^j m^{i+j-k-l} (2m-1)^k (n-1+m)^l + \sum_{i,j,k,l} {}_F c_{i,j,k,l} \cdot u^i v^{j+1} n^{i+j-k-l} (2n-1)^l (m-1+n)^k$$

The solution to the general Abel equation satisfies

$$\frac{n}{\alpha+n} \cdot \ln\left(F(\left(\frac{\alpha+n}{n}\right)^n \cdot g_1(x), \ldots, \left(\frac{\alpha+n}{n}\right)^n \cdot g_K(x), \frac{n \cdot (m_1 + \alpha)}{n+\alpha}, \ldots, \frac{n \cdot (m_K + \alpha)}{n+\alpha}) \right)$$

$$= \ln(F(g_1(x), \ldots, g_K(x), m_1, \ldots, m_K))$$

Define $a_k \equiv \left(\frac{\alpha+n}{n}\right)^n \cdot g_k$ and $b_k \equiv \frac{n \cdot (m_k + \alpha)}{n+\alpha}$. So

$\frac{n}{\alpha+n} \cdot \ln\left(F(a_1, \ldots, a_K, b_1, \ldots, b_K)\right) = \ln(F(g_1, \ldots, g_K, m_1, \ldots, m_K))$. Differentiate this equation with respect to each of $g_1, \ldots, g_K, m_1, \ldots, m_K$ in turn. We get for each $k \in [K]$,

$$\frac{n}{\alpha+n} \cdot \frac{F_{a,k}(a_1, \ldots, a_K, b_1, \ldots, b_K)}{F(a_1, \ldots, a_K, b_1, \ldots, b_K)} \cdot \left(\frac{\alpha+n}{n}\right)^n = \frac{F_{g,k}(g_1, \ldots, g_K, m_1, \ldots, m_K)}{F(g_1, \ldots, g_K, m_1, \ldots, m_K)}$$ and

$$\frac{n}{\alpha+n} \cdot \frac{F_{b,k}(a_1, \ldots, a_K, b_1, \ldots, b_K)}{F(a_1, \ldots, a_K, b_1, \ldots, b_K)} \cdot \left(\frac{n}{\alpha+n}\right) = \frac{F_{m,k}(g_1, \ldots, g_K, m_1, \ldots, m_K)}{F(g_1, \ldots, g_K, m_1, \ldots, m_K)}$$

where the pair of subscripts g,k means "partial differentiation with respect to the variable g_k", and m,k means "partial differentiation with respect to variable m_k", and a,k means "partial differentiation with respect to the variable a_k", and b,k means "partial differentiation with respect to the variable b_k". Define

$${}_{g,k}J(g_1, \ldots, g_K, m_1, \ldots, m_K) \equiv \frac{F_{g,k}(g_1, \ldots, g_K, m_1, \ldots, m_K)}{F(g_1, \ldots, g_K, m_1, \ldots, m_K)}$$ and

$${}_{m,k}J(g_1, \ldots, g_K, m_1, \ldots, m_K) \equiv \frac{F_{m,k}(g_1, \ldots, g_K, m_1, \ldots, m_K)}{F(g_1, \ldots, g_K, m_1, \ldots, m_K)}.$$

Under the transformation which simultaneously replaces all g_k with a_k and all m_k with b_k, ${}_{g,k}J$ becomes ${}_{a,k}J \cdot \left(\frac{\alpha+n}{n}\right)^{n-1}$ and ${}_{m,k}J$ becomes ${}_{b,k}J \cdot \left(\frac{n}{\alpha+n}\right)^2$.

Now, differentiate ${}_{a,k}J\cdot\left(\frac{\alpha+n}{n}\right)^{n-1} = {}_{g,k}J$ with respect to α. Get

$$\left(\frac{\alpha+n}{n}\right)^{n-1}\cdot\left\{\sum_{l=1}^{K}{}_{a,k}J_{a,l}\cdot\frac{\partial a_l}{\partial\alpha}+\sum_{l=1}^{K}{}_{a,k}J_{b,l}\cdot\frac{\partial b_l}{\partial\alpha}\right\}+(n-1)\cdot\frac{1}{n}\,{}_{a,k}J\cdot\left(\frac{\alpha+n}{n}\right)^{n-2}=0,\text{ or}$$

$$\left\{\left(\frac{\alpha+n}{n}\right)^{n}\cdot\sum_{l=1}^{K}{}_{a,k}J_{a,l}\cdot g_l+(\alpha+n)\cdot n\cdot\sum_{l=1}^{K}{}_{a,k}J_{b,l}\cdot\frac{n-m_l}{(n+\alpha)^2}\right\}+(n-1)\,{}_{a,k}J=0$$

since $b_k = n+n\cdot\frac{m_k-n}{n+\alpha}\Rightarrow\frac{\partial b_k}{\partial\alpha}=n\cdot\frac{n-m_k}{(n+\alpha)^2}$. Specialize $\alpha\to 0$ to get

$$\left\{\sum_{l=1}^{K}{}_{g,k}J_{g,l}\cdot g_l+\sum_{l=1}^{K}{}_{g,k}J_{m,l}\cdot(n-m_l)\right\}+(n-1)\,{}_{g,k}J=0.$$

Differentiate ${}_{b,k}J\cdot\left(\frac{n}{\alpha+n}\right)^{2}={}_{m,k}J$ with respect to α. Get

$$\left(\frac{n}{\alpha+n}\right)^2\cdot\left\{\left(\frac{\alpha+n}{n}\right)^{n-1}\cdot\frac{1}{n}\cdot\sum_{l=1}^{K}{}_{b,k}J_{a,l}\cdot g_l+n\cdot\sum_{l=1}^{K}{}_{b,k}J_{b,l}\cdot\frac{n-m_l}{(n+\alpha)^2}\right\}-2\frac{n^2}{(\alpha+n)^3}\,{}_{b,k}J=0$$

$$\left\{\left(\frac{\alpha+n}{n}\right)^{n}\sum_{l=1}^{K}{}_{b,k}J_{a,l}\cdot g_l+(\alpha+n)\cdot n\cdot\sum_{l=1}^{K}{}_{b,k}J_{b,l}\cdot\frac{n-m_l}{(n+\alpha)^2}\right\}-2\cdot{}_{b,k}J=0$$. Specialize $\alpha\to 0$ to get $\left\{\sum_{l=1}^{K}{}_{m,k}J_{g,l}\cdot g_l+\sum_{l=1}^{K}{}_{m,k}J_{m,l}\cdot(n-m_l)\right\}-2\cdot{}_{m,k}J=0$.

We seek solutions ${}_{g,k}J$ and ${}_{m,k}J$ of these two *linear partial differential equations* (LPDEs) which satisfy the additional conditions ${}_{g,k}J_{m,l}={}_{m,l}J_{g,k}$ and ${}_{g,k}J_{g,l}={}_{g,l}J_{g,k}$ and ${}_{m,k}J_{m,l}={}_{m,l}J_{m,k}$ for all $k,l\in[K]$.

First-order polynomial ordinary differential equations

Let $\sum_{i,j}g_{i,j}(x)\cdot z^{m_i}\cdot\left(\frac{dz}{dx}\right)^{n_j}=0$ have the solution $z=F(\vec{g},\vec{m},\vec{n})$. Then $\sum_{i,j}g_{i,j}(x)\cdot z^{m_i-\beta\cdot n_j}\cdot\left(\frac{z^{\beta}dz}{dx}\right)^{n_j}=0$ or $\sum_{i,j}g_{i,j}(x)\cdot z^{m_i-\beta\cdot n_j}\cdot\left(\frac{1}{\beta+1}\frac{dz^{\beta+1}}{dx}\right)^{n_j}=0$ or $\sum_{i,j}\frac{g_{i,j}}{(\beta+1)^{n_j}}\cdot u^{\frac{m_i-\beta\cdot n_j}{1+\beta}}\cdot\left(\frac{du}{dx}\right)^{n_j}=0$ where $u\equiv z^{\beta+1}$. So $u=F(\frac{g_{i,j}}{(\beta+1)^{n_j}},\frac{m_i-\beta\cdot n_j}{1+\beta},\vec{n})$. So $F(g_{i,j},m_i,n_j)=\left(F(\frac{g_{i,j}}{(\beta+1)^{n_j}},\frac{m_i-\beta\cdot n_j}{1+\beta},n_j)\right)^{\frac{1}{1+\beta}}$.

Chemical Kinetics

Let $\frac{dz_i}{dt} = \sum_{k=1}^{K} c_{i,k} \cdot \prod_{j=1}^{J} z_j^{L_{j,k}}$ for each $i \in [I]$. Let $\vec{z} = \vec{F}(\vec{c})$ denote the vector-valued function $z_i = F_i(\{\{c_{i,k}\}_{k=1}^{K}\}, \{L_{j,k})$. So $z_i^{v_i} \cdot \frac{dz_i}{dt} = \sum_{k=1}^{K} c_{i,k} \cdot \prod_{j=1}^{J} z_j^{L_{j,k}+v_i \cdot \delta_{i,j}}$. So $\frac{dz_i^{v_i+1}}{dt} = \sum_{k=1}^{K} (v_i+1) \cdot c_{i,k} \cdot \prod_{j=1}^{J} z_j^{L_{j,k}+v_i \cdot \delta_{i,j}}$. Define $u_i \equiv z_i^{v_i+1}$. Then $z_j = u_j^{1/(v_j+1)}$. So $\frac{du_i}{dt} = \sum_{k=1}^{K} (v_i+1) \cdot c_{i,k} \cdot \prod_{j=1}^{J} u_j^{\frac{L_{j,k}+v_i \cdot \delta_{i,j}}{1+v_j}}$. Crucial question: does the set $\{\frac{L_{j,k}+v_i \cdot \delta_{i,j}}{1+v_j}\}_{\substack{j\in[J]\\ k\in[K]}}$ consist of at most $J \cdot K$ values, for a given vector $\vec{v} \equiv (v_i)_{i=1}^{I}$? If so, then $u_i = F_i(\{\{(1+v_i) \cdot c_{i,k}\}_{k=1}^{K}\}, \{\frac{L_{j,k}+v_i \cdot \delta_{i,j}}{1+v_j}\}_{\substack{j\in[J]\\ k\in[K]}})$ for each $i \in [I]$