STABILITY OF PLASMA CONFIGURATIONS DURING COMPRESSION

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ABSTRACT

Magnetized Target Fusion (MTF) efforts are based on calculations showing that the addition of a closed magnetic field configuration reduces the driver pressure and rise time requirements for inertial confinement fusion by reducing thermal conductivity. However, instabilities that result in convective bulk transport at the Alfven time scale are still of concern since they are much faster than the implosion time. Such instabilities may occur during compression due to, for example, an increase in the plasma/magnetic pressure ratio β or, as in the case of a rotating plasma, spin-up due to angular momentum conservation. Details depend on the magnetic field and compression geometries.

A hard-core z pinch with purely azimuthal magnetic field can theoretically be made that relaxes into a wall supported diffuse profile satisfying the Kadomtsev criterion for the stability of $m = 0$ modes, which is preserved during cylindrical compression by an imploding outer return conductor. β at the center conductor surface must also be low enough to stabilize modes with $m \geq 1$. The stability of $m \geq 1$ modes actually improves during compression. A disadvantage of this geometry, though, is plasma contact with the solid boundaries. In addition to the risk of high atomic number impurity contamination during the initial turbulent bulk relaxation process, such contact causes plasma pressure to drop near the outer conductor, violating the Kadomtsev $m = 0$ stability criterion locally. The resultant instability can then convect impurities inward. Meanwhile, the center conductor (which is not part of the Kadomtsev profile) can go $m = 0$ unstable, convecting impurities outward. One way to mitigate impurity convection is to instead use a Woltjer-Taylor minimum magnetic energy configuration (spheromak). The sheared magnetic field inhibits convection, and the need for a center conductor is eliminated. The plasma, however, would likely still need to be wall supported due to unfavorable β scaling during quasispherical (3-D) compression.

Use of a Field Reversed Configuration (FRC) substantially resolves the wall contact issue, but at the cost of introducing a new (rotational) instability. An FRC has an open magnetic field outside its separatrix which effectively diverts wall material. However, FRC ions migrating outward across the separatrix have a nonzero average angular momentum, causing the FRC within to counter-rotate in response. When the FRC's rotational/diamagnetic drift frequency ratio α reaches a critical value of order unity, the FRC undergoes a rotational instability resulting in rapid disintegration. The instability is exacerbated by cylindrical compression since $\alpha \sim R^{-2/5}$ during this phase. A multipole magnetic field frozen into the outer conductor during compression may stabilize this mode directly and/or by impeding spin-up without significantly perturbing the implosion's azimuthal symmetry.

HARD-CORE z PINCH

The density ρ, magnetic field magnitude B, pressure p, and $\beta = 2\mu_0\pi p/B^2$ of an adiabatically compressed ideal Magnetohydrodynamic (MHD) z pinch with purely azimuthal magnetic field may be determined from the radius $r = r(r_0)$ of fluid elements initially at radii r_0 (a Lagrangian parameterization) [1],

$$\begin{array}{ll} \rho = \rho_0 \left(\frac{r}{r_0}\frac{dr}{dr_0}\right)^{-1} & B = B_0 \left(\frac{dr}{dr_0}\right)^{-1} \\ p = p_0 \left(\frac{r}{r_0}\frac{dr}{dr_0}\right)^{-5/3} & \beta = \beta_0 \left(\frac{r_0}{r}\right)^2 \left(\frac{r}{r_0}\frac{dr}{dr_0}\right)^{1/3} \end{array} \tag{1}$$

The "0" subscript identifies an initial profile property here and elsewhere, except for Q_0 and μ_0. If the compression occurs slowly enough that the plasma approximates an equilibrium at all times, r satisfies[1]

$$\frac{d}{dr_0}\left[p_0\left(\frac{r}{r_0}\frac{dr}{dr_0}\right)^{-5/3}\right] + \frac{1}{2\mu_0 r^2}\frac{d}{dr_0}\left[(rB_0)^2\left(\frac{dr}{dr_0}\right)^{-2}\right] = 0 \tag{2}$$

This ordinary differential equation (ODE) may be rewritten in terms of $u(r_0) = r^2$, and solved for $u'' = d^2u/dr_0^2$. For a hard-core z pinch with constant inner radius a, variable outer radius b, and an arbitrary initial equilibrium state, $u(r_0)$ may be evaluated numerically given boundary conditions $u(a) = a^2$, and some value of $u' = du/dr_0$ at $r_0 = a$. From this, one then finds $u''(a)$ from the ODE. $u(a+\Delta r_0)$ and $u'(a+\Delta r_0)$ may then be parabolically and linearly extrapolated, respectively, where Δr_0 is a small increment. Plugging these back into the ODE, one then finds $u''(a+\Delta r_0)$, from which one may proceed to extrapolate $u(a+2\Delta r_0)$ and $u'(a+2\Delta r_0)$ as before. The process is repeated until one reaches $r_0 = b_0$, at which time the value of $b = (u(b_0))^{1/2}$ for the solution is determined *ex post facto*. Solutions for various values of b are found by varying the initial choice of $u'(a)$.

We can then evaluate the stability of all azimuthal modes $\{m\}$ of the compressed state from Eqs. 1 using the Kadomtsev stability criteria[2],

$$\begin{array}{ll} Q_0 = \frac{-(6+5\beta)}{20}\frac{r}{p}\frac{dp}{dr} \leq 1 & (m=0) \\ Q_1 = -\beta\frac{r}{p}\frac{dp}{dr} \leq 1 & (m \geq 1) \end{array} \tag{3}$$

The above procedure is necessary to determine the compressed state properties in general, although $Q_0 \leq 1$ need only be satisfied initially for that to be the case thereafter[1]. We now focus on the special case of an initial "Kadomtsev profile" which, except for the numerical evaluation a two key parameters, has an analytic solution. This profile, as referred to here, has $Q_0 = 1$ everywhere, marginally satisfies the Kadomtsev $m = 0$ stability criterion. This profile is of special interest because a hard-core z pinch can theoretically be made that spontaneously relaxes roughly into one[3]. Given this for our initial state, we have the following properties parameterized by β_0,

$$\begin{array}{llll} \frac{p_0}{p_{a0}} = \frac{\beta_0^{5/2}(4+5\beta_{a0})^{5/2}}{\beta_{a0}^{5/2}(4+5\beta_0)^{5/2}} & \frac{r_0}{a} = \frac{\beta_{a0}^{3/4}(4+5\beta_0)^{1/4}}{\beta_0^{3/4}(4+5\beta_{a0})^{1/4}} \\ \frac{B_0}{B_{a0}} = \frac{\beta_0^{3/4}(4+5\beta_{a0})^{5/4}}{\beta_{a0}^{3/4}(4+5\beta_0)^{5/4}} & Q_1 = \frac{20\beta_0(5\beta_{0a}+4)^{5/2}}{(5\beta_0+6)(5\beta_0+4)^{5/2}} \end{array} \tag{4}$$

Here and elsewhere, subscripts "a" and "b" identify properties at $r = a$ and $r = b$, respectively. Q_1 falls off monotonically from its maximum at $r_0 = a$. The configuration, therefore, is stable to $m \geq 1$ modes if $Q_1 \leq 1$ at $r_0 = a$. This is equivalent to $\beta_{a0} \leq 2/5$.

The time invariant mass and magnetic flux per unit axial length within a fluid element of a general ideal MHD z pinch bounded by r_0 and r_0+dr_0 during compression are $2\pi\, r_0\rho dr_0$ and Bdr_0, respectively. The ratio, then, is a time invariant of the fluid element (parameterized by r_0) called the Kadomtsev parameter $K = K(r_0)$. Expressing it in terms of p instead of ρ via the ideal gas adiabatic invariant $p\rho^{-5/3}$, we have

$$K(r_0) = \frac{p^{3/5} r}{B} \tag{5}$$

Substituting in Eqs. 4, one finds that $K(r_0)$ is uniform for a Kadomtsev profile. When adiabatically compressed, then, K is a constant of both time and space for this profile,

$$\frac{p^{3/5} r}{B} = \frac{p_{a0}^{3/5} a}{B_{a0}} \tag{6}$$

Since the plasma is assumed to remain in equilibrium, Eq. 6 and the definition of β may be used to express the pressure balance equation

$$\frac{dp}{dr} + \frac{1}{2\mu_0 r^2}\frac{d}{dr}(rB)^2 = 0 \tag{7}$$

as a differential equation in p with reference to β, with the solution being a Kadomtsev profile. That is, uniformity of $p^{3/5} r/B$ in equilibrium implies and is implied by (is equivalent to) a Kadomstev profile, and the plasma remains in one during compression. So, $m = 0$ modes remain marginally stable, and we have

$$\begin{array}{llll} \frac{p}{p_a} = \frac{\beta^{5/2}(4+5\beta_a)^{5/2}}{\beta_a^{5/2}(4+5\beta)^{5/2}} & \frac{r}{a} = \frac{\beta_a^{3/4}(4+5\beta)^{1/4}}{\beta^{3/4}(4+5\beta_a)^{1/4}} \\ \frac{B}{B_a} = \frac{\beta^{3/4}(4+5\beta_a)^{5/4}}{\beta_a^{3/4}(4+5\beta)^{5/4}} & Q_1 = \frac{20\beta(5\beta_a+4)^{5/2}}{(5\beta+6)(5\beta+4)^{5/2}} \end{array} \tag{8}$$

Solutions to β_a, B_a, p_a, Q_1, and other properties such as ρ and T needed to complete the description and evaluate the stability of the compressed state are found first by choosing a value of β_a and then parameterizing the other properties in terms of it. One finds from Eq. 6 solved at $r = a$ and the definition of β, for example,

$$\frac{B_a}{B_{a0}} = \left(\frac{\beta_{a0}}{\beta_a}\right)^3 \qquad \frac{p_a}{p_{a0}} = \left(\frac{\beta_{a0}}{\beta_a}\right)^5 \tag{9}$$

Since compression causes B_a and p_a to increase, we see from these that $\beta_a \leq \beta_{a0}$. Since β_a

≤ 2/5 is the stability criterion for m ≥ 1 modes, a Kadomtsev profile initially stable to these modes remains stable during compression. All modes, then, remain stable.

The value of b for this solution, as was the case with the general solution to Eq. 2, is found *ex post facto*. From the Kadomtsev r profile of Eqs. 8,

$$\frac{b}{a}=\frac{\beta_a^{3/4}\left(4+5\beta_b\right)^{1/4}}{\beta_b^{3/4}\left(4+5\beta_a\right)^{1/4}} \tag{10}$$

For this we need β_b in addition to parameter β_a. To this end, we derive a relationship between a given fluid element's β and its initial value β_0 from the fact that flux interior to that element is preserved. This implies

$$\int_{\beta_a}^{\beta} B\frac{dr}{d\beta}d\beta=\int_{\beta_{a0}}^{\beta_0} B_0\frac{dr_0}{d\beta_0}d\beta_0 \tag{11}$$

Substituting the initial and compressed Kadomstev profiles (Eqs. 4 and Eqs. 8), and making use of Eq. 6 and the definition of β, we get

$$\frac{\beta_{a0}^3(5\beta_a+4)}{\beta_a^3(5\beta_{a0}+4)}\left(\frac{\ln\left(\frac{\beta(5\beta_a+4)}{\beta_a(5\beta+4)}\right)+20(\beta-\beta_a)}{\ln\left(\frac{\beta_0(5\beta_{a0}+4)}{\beta_{a0}(5\beta_0+4)}\right)+20(\beta_0-\beta_{a0})}\right)=1 \tag{12}$$

This may be solved numerically for $\beta(\beta_0)$, given β_{a0} and β_a. The endpoint β_{b0} (needed to find $\beta_b = \beta(\beta_{b0})$ for use in Eq. 10 to determine b) is determined by numerically solving for it in the initial Kadomstev r profile (Eqs. 4) evaluated at $r = b_0$,

$$\frac{b_0}{a}=\frac{\beta_{a0}^{3/4}\left(4+5\beta_{b0}\right)^{1/4}}{\beta_{b0}^{3/4}\left(4+5\beta_{a0}\right)^{1/4}} \tag{13}$$

The general $\beta(\beta_0)$ solution, meanwhile, allows one to calculate $\rho(\beta_0)$ and $T(\beta_0)$ from their initial profiles (Eqs. 8) via the ideal gas adiabatic invariant,

$$\frac{\rho\left(\beta_0\right)}{\rho_0\left(\beta_0\right)}=\left(\frac{p\left(\beta\left(\beta_0\right)\right)}{p_0\left(\beta_0\right)}\right)^{3/5} \qquad \frac{T\left(\beta_0\right)}{T_0\left(\beta_0\right)}=\left(\frac{p\left(\beta\left(\beta_0\right)\right)}{p_0\left(\beta_0\right)}\right)^{2/5} \tag{14}$$

Listed next are the results of a Kadomtsev profile marginally stable initially to all modes (β_{a0} = 2/5) with b_0/a = 10 compressed to the point where β_a = 0.17 (subsequently determined to correspond to b/a = 1.2). This example was chosen to match the parameters of a published MHD simulation intended to represent this case[1],

$$\begin{array}{lll} \frac{b_0}{a}=10 & \beta_{a0}=0.4 & \beta_a=0.17 \\ \frac{B_a}{B_{a0}}=13.03 & \frac{B_b}{B_{b0}}=76.77 & Q_{a1}=0.496 \\ \frac{p_a}{p_{a0}}=72.12 & \frac{p_b}{p_{b0}}=4.733\times10^4 & \frac{b}{a}=1.203 \\ \frac{\rho_a}{\rho_{a0}}=13.03 & \frac{\rho_b}{\rho_{b0}}=638.4 & \beta_{b0}=1.633\times10^{-2} \\ \frac{T_a}{T_{ao}}=5.536 & \frac{T_b}{T_{b0}}=74.14 & \beta_b=0.1311 \end{array} \tag{15}$$

These results are consistent with the reference's Fig. 5. One noteworthy feature, in addition to those discussed, is the much greater compression and heating at larger radii. In simulations which include thermal conductivity, contact with the outer conductor causes plasma pressure near the surface to drop, violating the Kadomtsev $m = 0$ stability criterion locally.[3] The resultant localized instability will then convect impurities inward, at least a small distance. Meanwhile, the center conductor (which is not part of the Kadomtsev profile) can go $m = 0$ unstable, convecting impurities outward. One open question, then, is the radial extent of transport from the unstable walls, as suggested in Fig. 1. The radiation cooling effect of such transport can be significantly mitigated if the walls are coated with Li or, better yet, frozen D_2.

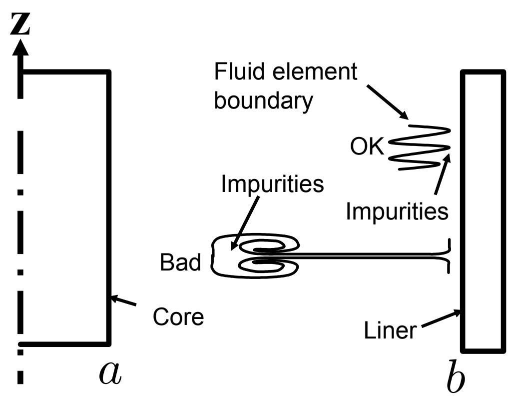

Figure 1. Hypothetical consequences of Kadomtsev unstable layer near wall.

A Woltjer-Taylor minimum magnetic energy configuration (spheromak) is an option that can reduce such transport. The sheared magnetic field inhibits convection, and the center conductor is eliminated. It is, however, only stable for low β if p at the wall is to be kept small. To make matters worse, $\beta \sim R^{-1}$ during quasispherical (3-D), where R is the liner radius. The plasma, therefore, would likely still have to be wall supported.

FIELD REVERSED CONFIGURATION

Using an FRC largely resolves the wall contact issue, but introduces a new instability. An FRC has open magnetic field lines outside a separatrix which effectively divert wall material. Ions migrating across the separatrix, though, have a net average angular momentum, causing the FRC within to counter-rotate in response.[4] When the FRC's rotational/diamagnetic drift frequency ratio α reaches a critical value of order unity, the FRC undergoes an instability with azimuthal mode number $n = 2$, resulting in rapid deconfinement. This instability can be understood conceptually and semi-quantitatively in terms of a simplified treatment that involves adding a finite Larmour radius stress tensor to the MHD equation of motion. Given this, Roberts and Taylor[5] show that, in planar geometry, an incompressible plasma whose initial density increases with height x as $\exp(\lambda x)$ in a gravitational field of acceleration g, Rayleigh-Taylor (R-T) modes are stable for magnetically transverse wave numbers k provided

$$g \le \nu^2 \lambda k^2 \qquad \nu = \frac{k_B T_i}{2ZeB} \tag{16}$$

Here, ν is the "gyroviscosity" coefficient, with k_B, T_i, Z, e, and B being the Boltzmann constant, ion temperature, mean ionization level, elementary charge, and magnetic field magnitude, respectively.

To apply this result to a FRC rotating with angular velocity Ω_R and an axial magnetic field of characteristic magnitude B, note that the ion diamagnetic drift frequency is

$$\Omega_{Di} = -\frac{v_{Di}}{r} \qquad \mathbf{v}_{Di} = -\frac{\boldsymbol{\nabla} p_i \times \mathbf{B}}{eZn_i B^2} \tag{17}$$

where $\mathbf{v}_{Di}$ is the diamagnetic drift velocity, and p_i and n_i are the ion pressure and number density, respectively. Using centripetal acceleration for g at characteristic radius R_1 (magnetic null radius), and wrapping the mode "plane" around the circumference, we have, then, the characteristic values

$$\Omega_{Di} = \frac{2\lambda\nu}{R_1} \qquad g = \Omega_R^2 R_1 \qquad k = \frac{n}{R_1} \tag{18}$$

Our stability criterion $g \le \nu^2\lambda k^2$ is, then,

$$\alpha = \frac{\Omega_R}{\Omega_{Di}} \le \frac{n}{2\sqrt{\lambda R_1}} \tag{19}$$

Given a density gradient scale length of $1/\lambda \approx R_1$, stability requires $\alpha \le 1$ for the least stable mode $n = 2$. Technically, $n = 1$ goes unstable first, but the (planar) model applied to cylindrical geometry does not conserve lateral linear momentum for this mode, so is inapplicable. The threshold is respectably close to the threshold value observed[6] and described by more sophisticated modeling[7-9], given the geometrical liberties taken.

The $n = 2$ mode is of particular concern for the MTF application because α theoretically increases significantly during wall compression by a conducting cylindrical liner. To show this, firstly, angular Ω_R increases in proportion to R_s^{-2} from angular momentum conservation, where R_s is the separatrix radius. Meanwhile, $x_s = R_s/R_c$ is conserved during cylindrical wall compression, where R_c is the liner inner radius (Tuszewski, p. 2058).[10] Given this, plasma β is conserved (Tuszewski, Eq. 10).[10] Given *this* and flux conservation, Ω_{Di} is proportional to T_i, (Shimamura, Eq. 7 with $\Omega^* = -\Omega_{Di}$).[11] The FRC's characteristic volume $V = \pi R_s^2 L_s$, meanwhile, decreases as R_s^N, where L_s is the separatrix length, and N is the dimensionality of compression. Assuming adiabatic compression, $T_i V^{(5/3)-1}$ is conserved. Therefore, $\Omega_{Di} R_s^{2N/3} =$ constant, and Ω_{Di} increases in proportion to $R_s^{-2N/3}$. α, then, increases in proportion to $R_s^{-2}/R_s^{-2N/3} = R_s^{-2(1-N/3)}$. $N = 12/5$ for cylindrical wall compression (Tuszewski, Table V)[10], so α increases in proportion to $R_s^{-2/5}$ or, equivalently, $R_c^{-2/5}$. The (target) factor of 10 radial compression, then, increases α by a factor of $10^{2/5} \approx 2.5$. A multipole magnetic field frozen into the solid liner during compression may stabilize this mode directly, and/or by impeding spin-up, without significantly perturbing the implosion's azimuthal symmetry.[12]

SUMMARY

The effects of compression on the most problematic instabilities of two leading candidates for MTF are discussed. For a hard-core z pinch, a slowly compressed initial Kadomtsev profile remains marginally stable to $m = 0$ modes during cylindrical radial compression in the ideal MHD limit, and $m \geq 1$ modes are over-stabilized. However, radial convection of wall material is expected to result from the destabilization of the profile due to wall interaction. The significance is yet to be determined, and mitigation strategies such as low Z wall loading are possible. For an FRC, disintegration during compression is expected due to unfavorable scaling of the rotational $n = 2$ mode's stability threshold parameter α. Magnetic multipole stabilization of this mode is a possible solution that needs further investigation.

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