The FRC's n=2 rotational instability interpreted as the dominant Rayleigh-Taylor mode of a gyroviscous plasma with sheared toroidal flow

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A Field Reversed Configuration is observed to gain angular momentum until $\alpha = \Omega_R/\Omega_{Di}$ (rotational frequency over ion diamagnetic drift frequency) reaches a critical value, at which point an instability with azimuthal mode number $n = 2$ develops. Questions remain as to whether the observed threshold is explained by published calculations, which assume a rigid rotor profile. Questions also remain as to the cause of the spin-up, but it necessarily involves angular momentum transport to the FRC through the outer surface. Rotation of the bulk, then, via kinematic viscosity and/or convection can result in significant velocity shear. Rotation results in plasma (centripetal) acceleration supported by an external magnetic field, so the instability may be interpreted as a Rayleigh-Taylor mode. Both sheared flow and Finite Larmour Radius effects are recognized as mitigating factors for the R–T instability, and the two effects are synergistic.

The rotational instability is investigated here using an analytic planar R-T model of an FLR plasma with a magnetically transverse sheared flow layer accelerated by the magnetic field. One result is that if the sheared layer is too thin to reach the magnetic (reversal) axis, it is unstable. The coupling between gyroviscosity and flow shear in this case negates the stabilizing effect of both within a range of modes, and convection of the sheared layer to the magnetic axis can be expected to occur quickly. Once this happens, though, the FRC is stable until the shear factor reaches a high value, at which time the $n = 2$ mode goes unstable. Technically, $n = 1$ goes unstable first, but the (planar) model applied to cylindrical geometry does not conserve momentum for this mode, so it is unphysical.

This model provides insight into what may be an important feature of FRC stability, although less simplified calculations are needed. Nonetheless, it can be used to tentatively predict stability characteristics of an FRC during compression by an imploding metal cylinder – a goal of the Magnetized Target Fusion program. This is of concern since acceleration from such an implosion supplements centripetal acceleration, and α more than doubles, assuming angular momentum conservation, adiabatic compression, and the expected volume vs. radius scaling.

Introduction

- FLR MHD [**Friedberg78**], Vlasov [**Seyler79**], and hybrid [**Harned83**] simulations of the $n = 2$ rotational instability in θ pinches and FRC's to date generally assume the plasma rotates with a rigid rotor profile.
- $\alpha \gtrsim 1$, ($\alpha = \Omega_R/\Omega_{Di}$ (rotational frequency over ion diamagnetic drift frequency) reasonably consistent what is observed for first two (θ pinches), but $\alpha \gtrsim 0.4$ implied for Hybrid (FRC simulation).
- Lower critical α attributed to resonant ions near the magnetic axis due to the magnetic field vanishing there - an effect not present in a θ pinch or representable by the FLR stress tensor.

Friedberg78 J. P. Freidberg and L. D. Pearlstein, Phys. Fluids **21**, 1207 (1978).

Seyler79 C. E. Selyer, Phys. Fluids **22**, 2324 (1978).

Harned83 D. S. Harned, Phys. Fluids **26**, 1320 (1981).

- The equilibration time scale for Ω_R is

$$\tau_\Omega = \frac{(r_s - r_0)^2}{\nu_k}$$

where r_s and r_0 are the separatrix and magnetic axis (where B_z reverses), respectively, ν_k is kinematic viscosity.

- For a D_2 plasma with $r_s - r_0 = 1$ cm, $T_i = T_e = 200$ eV, and $\ln\Lambda \approx 15$, typical of MTF FRC's [**Intrator04b**], $\tau_\Omega \approx 83\ \mu$s. An $n = 2$ instability, though, is observed to develop within 15 μs of formation.

Intrator04b T. Intrator, S. Y. Zhang, J. H. Degnan, I. Furno, C. Grabowski, S. C. Hsu, E. L. Ruden, P. G. Sanchez, J. M. Taccetti, M. Tuszewski, W. J. Waganaar, and G. A. Wurden, Phys. Plasmas **11**, 2580 (2004).

- A mechanism is proposed whereby a sheared flow layer establishes itself between r_s and r_0 on an MHD time scale by convection resulting from the instability of the sheared layer while it is thinner.
- Once that occurs, a much more stable shear flow pattern emerges until Ω_R reaches a critical value.
- Plasma with $r < r_0$ connected to field lines within the sheared layer will be carried along at the same Ω_R, but is R-T stable.

- $n = 2$ is of particular concern for MTF because α increases significantly during wall compression by a cylindrical liner.
- FRC volume $V = \pi r_s^2 l_s$, where l_s is the separatrix length, goes as r_s^N, where N is the dimensionality of compression.
- Assuming expected r_s and l_s scaling (**Tuszewski88**, Table V) with flux conservation and adiabatic compression by a cylindrical wall, α goes as $r_c^{-2(1-N/3)}$, where r_c is the liner inner radius.
- $N = 12/5$ for wall compression so $10 \times$ radial compression increases α by a factor of $10^{2/5} \approx 2.5$.

Tuszewski88 M. Tuszewski, Nuclear Fusion **28**, 2033 (1988).

Model

The model we use to study the development of the FRC's sheared flow [**ruden04**] is planar with **x**, **y**, and **z** directions corresponding to $-\mathbf{r}$, $\boldsymbol{\theta}$, and **z** of the FRC's cylindrical coordinates, respectively. The reference frame is be rotating, with centripetal acceleration providing "gravity" g.

ruden04 E. L. Ruden, Phys. Plasmas **11**, 713 (2004).

We assume a perfectly conducting isothermal z invariant plasma with a magnetic field of magnitude B in the unit vector $\mathbf{z}$ direction, and a uniform gravitational field of magnitude g in the $-\mathbf{x}$ direction. For incompressible motion confined to the $x - y$ plane, the MHD equation of motion supplemented by the isothermal transverse contribution to the FLR stress tensor $\mathbf{\Pi}$ [**HM92** Chap. 6, Eq. 123], along with the equations of continuity and state are assumed

HM92 R. D. Hazeltine and J. D. Meiss, *Plasma Confinement* (Addison-Wesley, Redwood, CA, 1992).

$$\rho \partial \mathbf{v}/\partial t + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p^* - g\rho \mathbf{x} - \nabla \cdot \mathbf{\Pi}$$

$$\partial \rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \nabla \cdot \mathbf{v} = 0 \quad T \equiv (T_i + ZT_e)$$

$$p^* \equiv k_B T \rho / m_i + B^2/(2\mu_0) \qquad \nu = k_B T_i/(2ZeB)$$

$$-(\nabla \cdot \mathbf{\Pi}) \cdot \mathbf{x} = \frac{\partial}{\partial x}\left[\nu\rho\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\right] - \frac{\partial}{\partial y}\left[\nu\rho\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)\right]$$

$$-(\nabla \cdot \mathbf{\Pi}) \cdot \mathbf{y} = -\frac{\partial}{\partial y}\left[\nu\rho\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\right] - \frac{\partial}{\partial x}\left[\nu\rho\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)\right]$$

$\mathbf{v}$, ρ, m_i, μ_0, k_B, e, and ν are the fluid velocity, density, ion mass, free space permeability, Boltzmann constant, elementary charge, and gyroviscosity coefficient, respectively.

The equilibrium states of interest have $\mathbf{v} = V(x)\mathbf{y}$, $\rho = \rho_0(x)$, $\mathbf{B} = B_0(x)\mathbf{z}$, and $\nu = \nu_0(x)$. For instabilities with short wavelengths relative to the sheared flow layer thickness, consider case I of two semi-infinite regions separated at $x = 0$ with equilibrium conditions

$$\rho_0 = \rho_1 \qquad V = s_1 x \qquad \nu_0 = \nu_1 \quad \text{if} \quad x < 0$$

$$\rho_0 = \rho_2 \qquad V = s_2 x \qquad \nu_0 = \nu_2 \quad \text{if} \quad x \geq 0$$

where ρ_1, ρ_2, s_1, s_2, ν_1, and ν_2 are constants.

The condition for linear instability of a perturbation with $\exp(i\omega t + iky)$ dependence, implying maximum growth rate $\gamma = \max(-\mathrm{Im}\,\omega)$, is

$$J^* > \frac{\left(1 - G^* K^2\right)^2}{2K} \rightarrow \Gamma = \sqrt{1 - \frac{\left(1 - G^* K^2\right)^2}{2J^* K}}$$

$$J^* \equiv \frac{g^*}{s^{*2} d} \qquad G^* \equiv \frac{\nu^*}{2s^* d^2} \qquad \Gamma \equiv \frac{\gamma}{\sqrt{g^* k}}$$

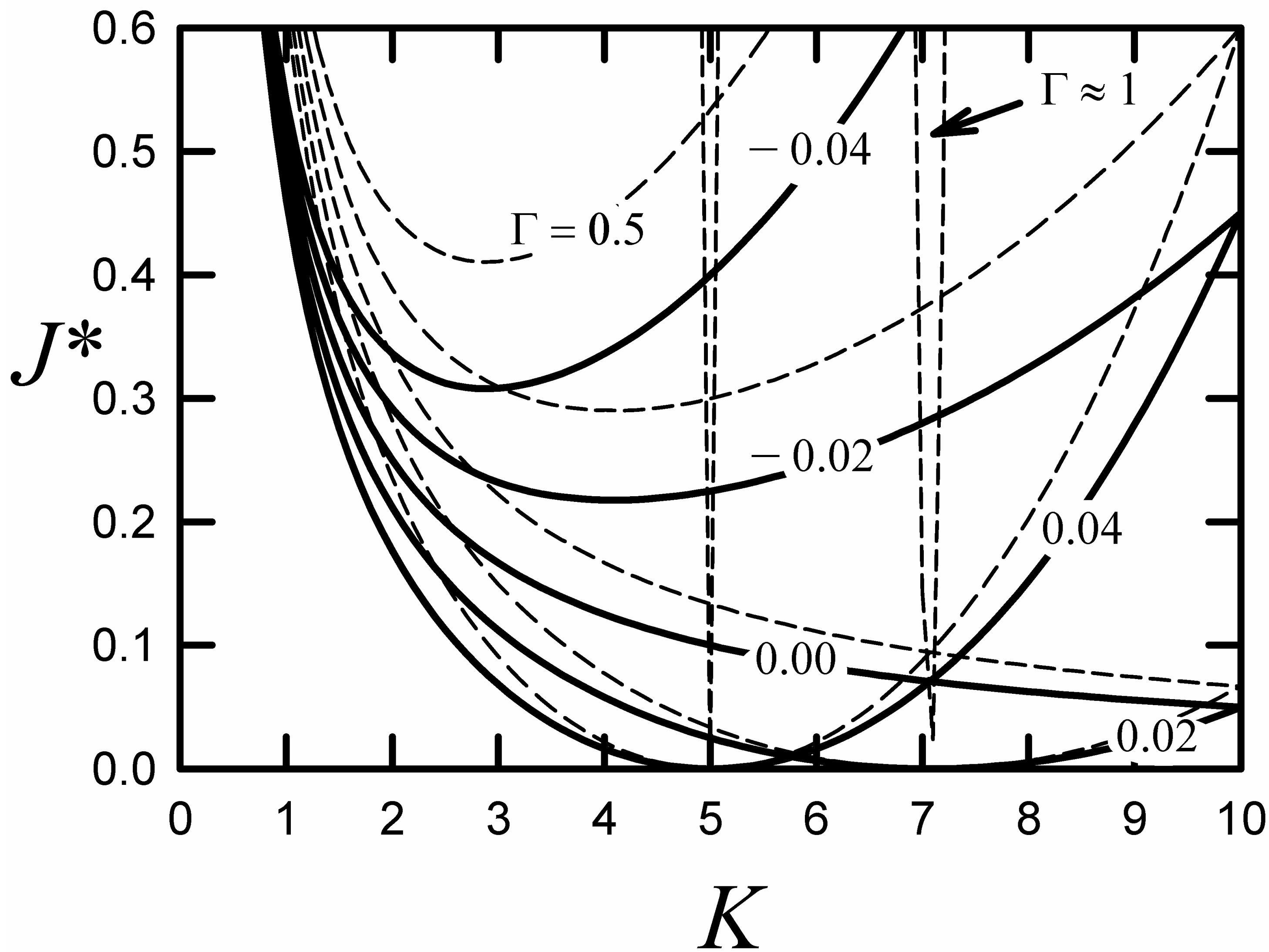
$$K \equiv 2kd \quad \nu^* = \frac{\nu_2 \rho_2 - \nu_1 \rho_1}{\rho_1 + \rho_2} \quad s^* = \frac{s_2 \rho_2 - s_1 \rho_1}{\rho_1 + \rho_2} \quad g^* = g \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}$$

Here, $2d$ is the thickness of the plasma layer intended to be modeled.

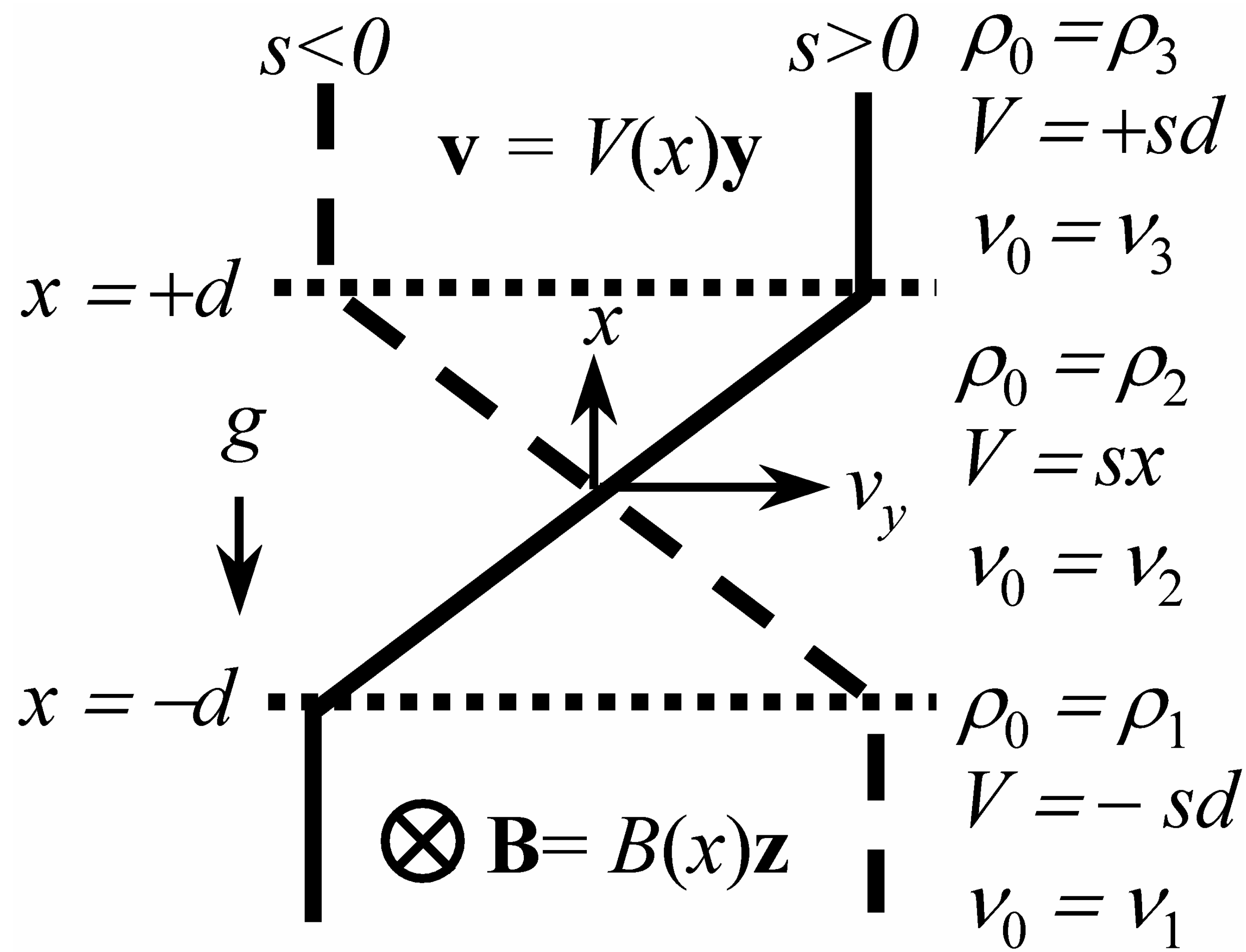
An R-T relevant configuration is one for which $\rho_2 > \rho_1$, implying positive J^*. G^* may be of either sign regardless of the relative magnitudes of ρ_1 and ρ_2. The stability condition for all K is

$$J^* \leq \begin{cases} 8\sqrt{-G^*/27} & \text{if } G^* < 0 \\ 0 & \text{if } G^* \geq 0 \end{cases}$$

Note from the above that an R-T relevant configuration is unstable if $G^* > 0$, and gyroviscosity suppresses the stabilizing influence of flow shear in the vicinity of $K = 1/\sqrt{G^*}$.



Case I stability boundary (solid) and associated $\Gamma = 0.5$ and 1.0 contours (dashed, near and above associated stability boundary) for labeled G^* values. Modes are stable below the respective curves. For $G^* > 0$, the stabilizing effect of flow shear is negated (ie. $\Gamma \approx 1$) for $K \approx 1/\sqrt{G^*}$.



Case II Geometry and coordinate system assumed - a flow sheared layer separating two unsheared semiinfinite regions. The $\otimes$ “tail feathers” symbol indicates the magnetic field direction is into the page. The solid/dashed V vs. x plot corresponds to a positive/negative shear factor s.

Case II

$$\rho_0 = \rho_1 \qquad V = -sd \qquad v_0 = v_1 \quad \text{if} \quad x < -d$$

$$\rho_0 = \rho_2 \qquad V = sx \qquad v_0 = v_2 \quad \text{if} \quad -d < x < +d$$

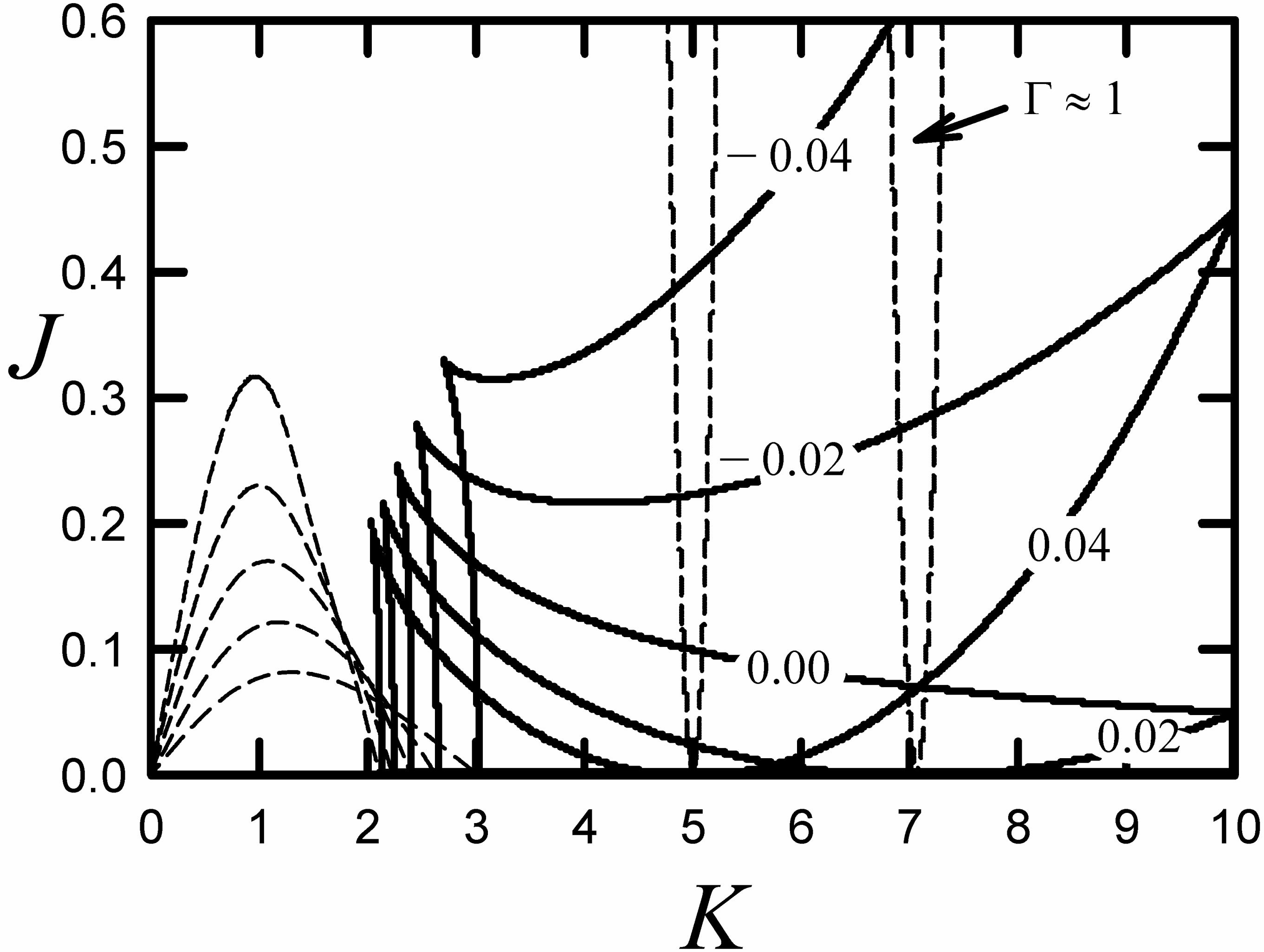
$$\rho_0 = \rho_3 \qquad V = +sd \qquad v_0 = v_3 \quad \text{if} \quad x > +d$$

$$\frac{(1-\epsilon_3)(1+\Omega_+^2)-2K(G_2-\epsilon_3 G_3)\Omega_0\Omega_+ + 2\Omega_0\Omega_+/K}{(1-\epsilon_3)-(1+\epsilon_3)\Omega_+^2-2K(G_2-\epsilon_3 G_3)\Omega_0\Omega_+ + 2\Omega_0\Omega_+/K} e^{-2K}$$

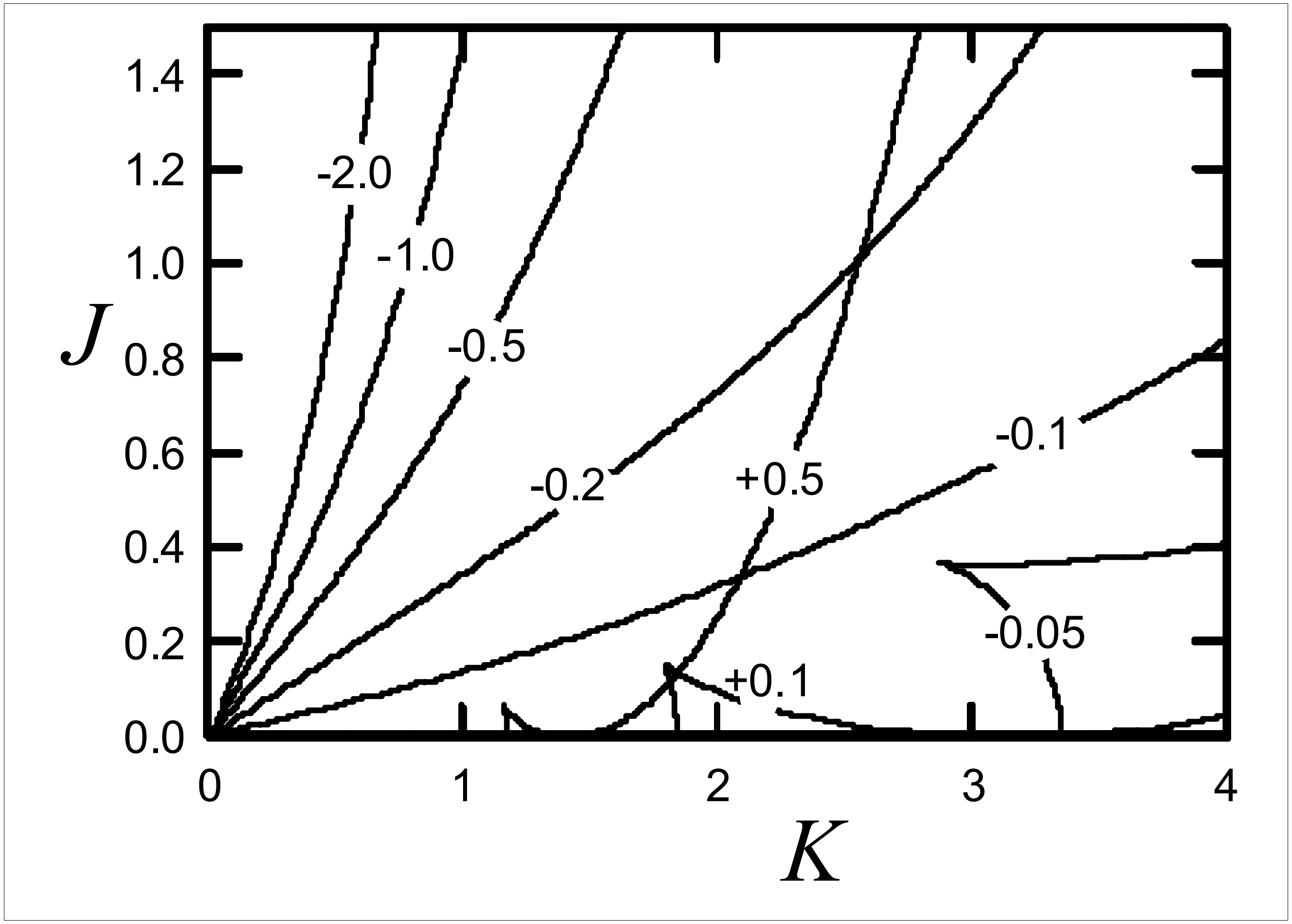
$$= \frac{(1-\epsilon_1)+(1+\epsilon_1)\Omega_-^2-2K(G_2-\epsilon_1 G_1)\Omega_0\Omega_- + 2\Omega_0\Omega_-/K}{(1-\epsilon_1)(1-\Omega_-^2)-2K(G_2-\epsilon_1 G_1)\Omega_0\Omega_- + 2\Omega_0\Omega_-/K}$$

$$\Omega \equiv \frac{\omega}{\sqrt{gk}} \qquad \Omega_0 \equiv \sqrt{\frac{K}{2J}} \qquad \Omega_\pm \equiv \Omega \pm \Omega_0$$

$$K \equiv 2kd \qquad J \equiv \frac{g}{s^2 d} \qquad G_i \equiv \frac{v_i}{2sd^2} \qquad \epsilon_i \equiv \frac{\rho_i}{\rho_2}$$



Stability boundaries (solid) and $\Gamma \approx 1$ contours (dashed) for Case II with $\epsilon_1 = \epsilon_3 = 0$, and the same values of G_2 as G^* in Case I plot. Note the close agreement with Case I for the high K.



Stability boundaries for case II with $\epsilon_1 = \epsilon_3 = 0$ for various *larger* values of G_2 relevant to FRC's. The stable region is to the right and below each contour. Note for $G_2 > 0$, the value of K unstable for all J is $K = 1/\sqrt{G_2}$, the same as with case I.

FRC application

We now relate this model to an FRC. Ion diamagnetic drift is

$$\Omega_{Di} = -\frac{v_{Di}}{r} \qquad \mathbf{v}_{Di} = -\frac{\nabla p_i \times \mathbf{B}}{eZn_i B^2}$$

where $\mathbf{v}_{Di}$ is diamagnetic drift velocity, p_i and n_i are ion pressure and number density, respectively, and we take the external magnetic field **B** to be in the **z** direction. The exterior pressure scale length of the FRC is $(r_s - r_0)$, so we take the characteristic Ω_{Di} to be,

$$\Omega_{Di} = -\frac{2Av}{r_0^2} \quad A = \frac{r_0}{r_s - r_0} \qquad r_s = r_0\frac{1+A}{A}$$

Where A is a measure of the FRC aspect ratio.

Ω_R is taken to be Ω at $r = r_s$, which drops to zero at $r = r_s - 2d$. We take our characteristic g to be half its peak value and, since g goes as Ω_R^2, we'll take the characteristic angular velocity (for the purposes of defining α) to be $\Omega_R/\sqrt{2}$. We have, then, the following case II parameters in terms of FRC parameters.

$$g = \frac{\Omega_R^2 r_s}{2} \qquad \alpha \equiv \frac{\Omega_R}{\sqrt{2}\,\Omega_{Di}} \qquad A^* = \frac{r_0}{2d}$$

$$\Omega_R = \frac{sA}{A^*(1+A)} \qquad G_2 \equiv \frac{\nu}{2sd^2} = -\frac{A^*}{\sqrt{2}\,\alpha(1+A)}$$

$$J = \frac{g}{s^2 d} = \frac{A}{A^*(1+A)} \qquad K = 2kd = \frac{n}{A^*}$$

A^* is the aspect ratio of the sheared layer. Ω_R develops with the same sign as Ω_{Di} [**Tuszewski90**] ($\alpha > 0$). Therefore, s and are negative.

Tuszewski90 M. Tuszewski, Phys. Fluids B **2**, 2541 (1990).

Thin shear layer

Applying case I, consider a shear layer represented by region 1 which has diffused a small distance into the FRC ($A^* \gg A$). The interior is taken to be region 2 with $s_2 = 0$ and a higher density of, say, $\rho_2 = 2\rho_1$. B is fairly uniform in the vicinity of interest, so $v_2 = v_1$ is assumed,

$$v^* = \frac{v_1}{2} \quad s^* = -\frac{s_1}{3} \quad g^* = \frac{g}{3} \quad J^* = 3\left(\frac{g}{s_1^2 d}\right) \quad G^* = -\frac{3}{2}\left(\frac{v_1}{2 s_1 d^2}\right)$$

So $G^* > 0$, and the configuration is unstable with least stable mode $K = 1/\sqrt{G^*}$ or,

$$n = \sqrt{\frac{2\sqrt{2}}{3}}\sqrt{\alpha A^*(1 + A)} \approx \sqrt{\alpha A^*(1 + A)}$$

For $A = 2$, $A^* = 10$, and $\alpha = 1$, the least stable mode is $n = 5$.

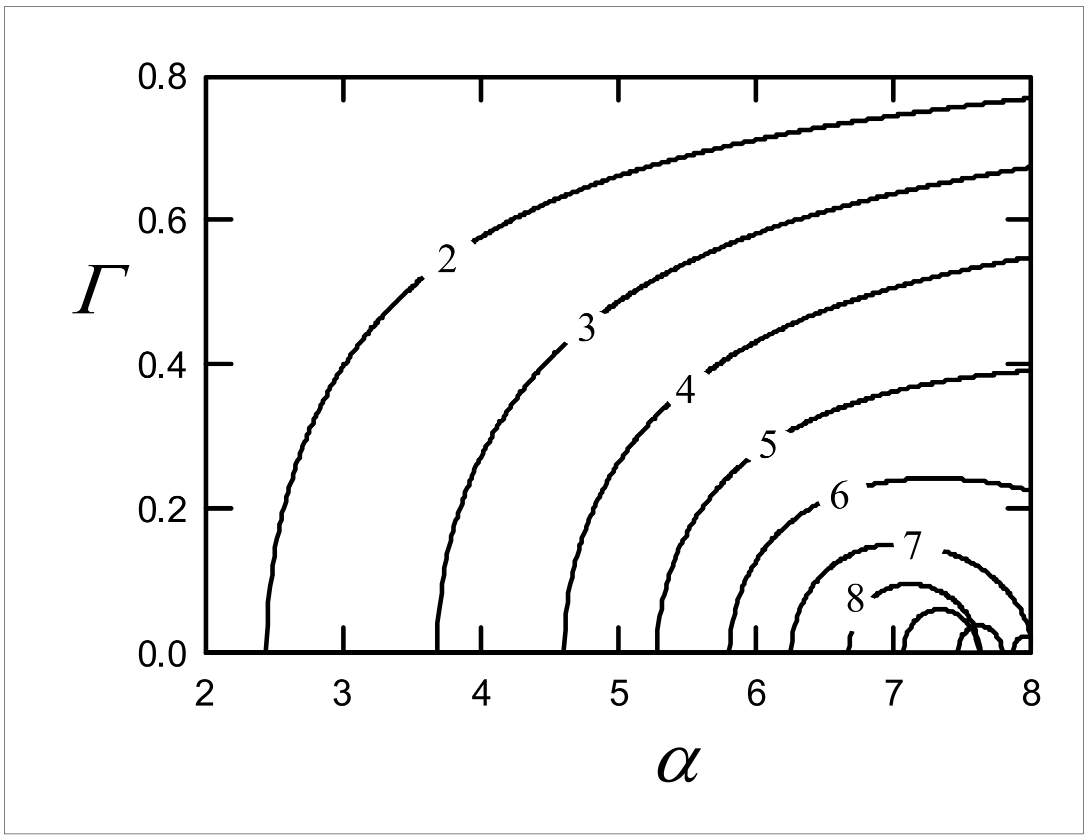
The FRC must survive this turbulent period where unstable modes become of increasingly lower order as the FRC settles down and A^* decreases. Convective transport of angular momentum toward the interior can be expected on time scales much shorter than can be accounted for by viscous laminar diffusion until the shear layer reaches $r = r_0$ $(A^* = A)$ where $G^* > 0$ is no longer the case.

Thick shear layer

Assuming a well sheared layer diffused to r_0 ($A^* = A$), case II parameters are,

$$G_2 = -\frac{A}{\sqrt{2}\,\alpha(1+A)} \qquad J = \frac{1}{(1+A)} \qquad K = \frac{n}{A}$$

This plot illustrates the growth rate vs. α for the first several modes with $A = 2$.



Conclusions

Substantially lower values of α are observed experimentally than can be explained by FLR MHD theory in the context of the presented model. One possible explanation is that resonant ions (not represented by FLR MHD) near the magnetic axis reduce stability well below the $\alpha = 2.5$ value of our model. If this is the case, it may be possible to substantially increase the critical α in FRC experiments if resonant ions can be damped by, for example, introduction of a small toroidal magnetic field