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Rayleigh-Taylor Stability Criteria for Magnetically Imploded Solid Liners

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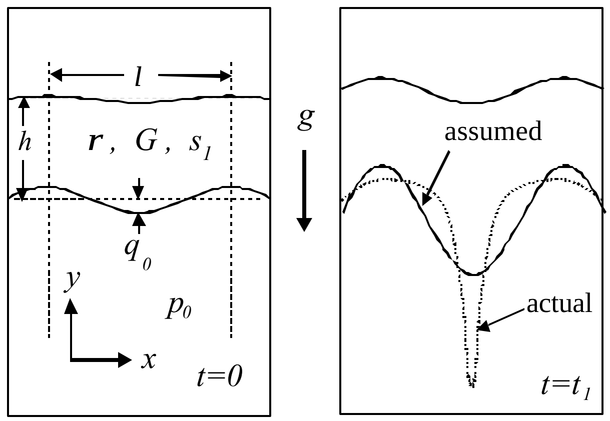
Abstract

Approximate analytic Rayleigh-Taylor stability criteria for elastic-plastic solid liners, where convergence imposes a background strain rate tensor [1], is reviewed and compared to other work. It is shown that in the incompressible two-dimensional planar limit, with constant yield strength and shear modulus, the dynamics of a liner driven by a massless fluid vs. a magnetic field orthogonal to the perturbation are equivalent. This holds regardless of the form or magnitude of the perturbation or degree of magnetic diffusion. The proposition that a redistribution of the magnetic pressure due to diffusion has no effect on the material dynamics has proven to be counterintuitive to many, so will be discussed in depth. Beyond the above limit, however, thermal softening and phase transitions due to Ohmic heating, compressibility, three-dimensional flow, and cylindrical geometry can be issues. The conditions where these effects can be expected to significantly effect the dynamics are discussed.

[1] E.L. Ruden and D.E. Bell, J. Appl. Phys. 82, 163-170 (1997)

I. Analytic Models



Simplifications for analytic treatment - acceleration g, density r, shear modulus G, von-Mises yield strength s_1, and pressure p_o all assumed constant. The ideal fluid linear eigenfunction (exponential-sinusoidal) is assumed for the spatial profile.

- **Governing Equations**:

Prandtl-Reuss elastic-plastic flow relating the deviator stress and strain rate tensors $\mathbf{S}$ and $\mathbf{D}$, respectively,

$$\dot{\mathbf{S}} + \Lambda \mathbf{S} = 2G\mathbf{D}, \tag{1}$$

where,

$$\mathbf{D} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \tag{2}$$

and,

$$\Lambda = \begin{cases} G\dot{W}/s_1^2 & \text{if } J_2 = 2s_1^2 \text{ and } \dot{W} > 0 \\ 0 & \text{if } J_2 < 2s_1^2 \text{ or } \dot{W} < 0 \end{cases} \tag{3}$$

$J_2 \equiv \mathbf{S} \cdot \mathbf{S}$ and is self-consistently bounded by $2s_1^2$ (von Mises yield condition). The equation of motion is,

$$\rho \dot{\mathbf{v}} = -\nabla p + \nabla \cdot \mathbf{S} - \rho g \hat{\mathbf{y}}, \tag{4}$$

and the (incompressible) equation of state is

$$\nabla \cdot \mathbf{v} = \mathbf{0} \tag{5}$$

- **Velocity profile of perturbation assumed**,

$$(v_x, v_y, v_z) = \dot{q}e^{-ky}(\sin kx, \cos kx, 0) \tag{6}$$

  Best used for determining stability criteria as opposed to unstable growth profile.

- Approximated equation of motion (EOM) of White 73 based on conservation of global balance of gravitational, kinetic, and plastic work energy assuming Prantl-Reuss rules for elastic plastic flow. Convergence/divergence geometry represented by parameter A.

$$A = 1 - [(S_{xx} - S_{yy})/2s_1]^2 + (S_{xy}/s_1)^2 \tag{7}$$

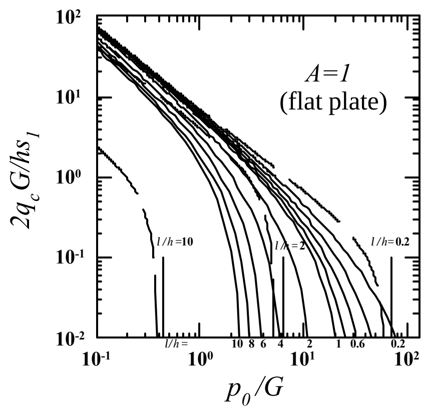
  where $\mathbf{S}$ is the *unperturbed* deviatoric stress tensor.

[G. N. White, "A One-Degree-of-Freedom Model for the Taylor Instability of an Ideally Plastic Metal Plate", LANL Tech Rep. LA-5225-MS , APR 1973]

- Stability threshold curves of White EOM derived in parametric form.

[E.L. Ruden and D.E. Bell, "Rayleigh-Taylor Stability Criteria for Elastic-Plastic Solid Plates and Shells, J. Appl. Phys. 82, 163-170 (1997)]

convergence	perturbation	(S_{xx}, S_{yy}, S_{zz})	A
cylindrical	azimuthal	$(-1, 1, 0)s_1$	0
spherical	any	$(-1, +2, -1)s_1/\sqrt{3}$	1/4
cylindrical	axial	$(0, 1, -1)s_1$	3/4
flat	any	$(0, 0, 0)$	1



Stability boundaries for Ruden 97 (thin lines) and Lebedev 96 (dashed lines). q_c is the maximum initial perturbation amplitute for stability. Vertical lines are critical pressure ($q_c \to 0$) based on Plohr 98 for the wavelengths for which Lebedev 96 is plotted.

- Significant descrepency seen between Ruden 97 and Lebedev 96 for large λ/h.

[A. I. Lebedev, et.al., "Rayeigh-Taylor Instability in Solids", Dolklady Akademii Nauk, **349**, 332-334, 1996]

- Lebedev 96 agrees closely with Plohr 98, which only considered stability of infinitesimal perturbation, but allows for vorticity.

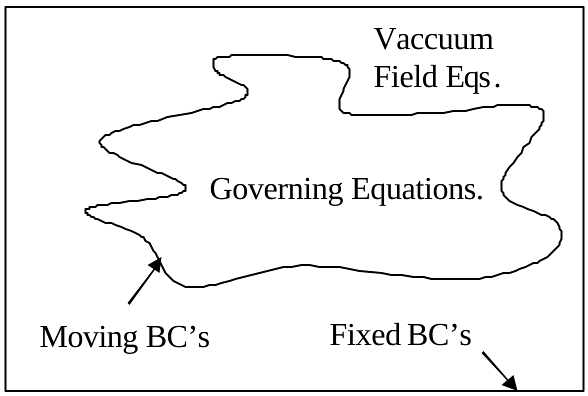
[B.J. Plohr and D.H. Sharp, Instability of accelerated elastic metal plates, Z. angew. Math. Phys., **49**, 786-804, 1998]

- Good correlations between Plohr 98 and experimental data shown in this paper. More rigorous perturbation theory used too.
- Vorticity neglected in the linear ideal R-T eigenfunction assumed in Ruden 97, so...
- Lebedev 96 probably best analytic model of R-T for finite initial perturbations. Results available in English as part of Bakhrakh 97, a massive compilation of Russian work on instability of solids. Recommended - Download from LLNL website.

[S. M. Bakhrakh, et.al. "Hydrodynamic Instability in Strong Media", LLNL TechRep. UCRL-CR-126710, March97]

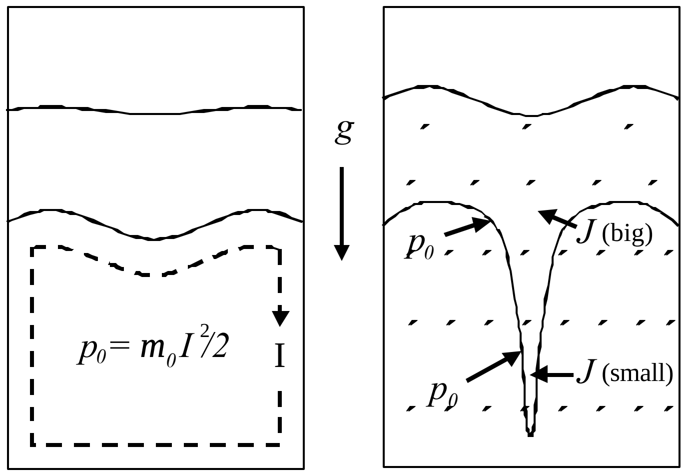
II. Effect of magnetic diffusion per se (with constant material properties)

- What differences can be expected if the driver is a magnetic field? First consider the general MHD problem of a continuous medium:



General MHD problem applied to a continuous medium. In special cases where a subset of the medium's governing equations and BC's are mathematically equivalent to a fully characterized hydrodynamic problem, The solution to that subset will, of course, be the same.

- Without making any simplifications on the velocity profile we have:



Corresponding magnetic driver case with strong diffusion. Planar limit, no field curvature or shear, 2-D perturbation orthogonal to **B**, and constant *r*, *G*, *Y*, and *I* (current per unit length) leads to same dynamics.

- **Governing equations:**

- Prandtl-Reuss elastic-plastic flow relating the deviator stress and strain rate tensors $\mathbf{S}$ and $\mathbf{D}$, respectively,

$$\dot{\mathbf{S}} + \Lambda \mathbf{S} = 2G\mathbf{D}, \tag{8}$$

where,

$$\mathbf{D} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \tag{9}$$

and,

$$\Lambda = \begin{cases} G\dot{W}/s_1^2 & \text{if } J_2 = 2s_1^2 \text{ and } \dot{W} > 0 \\ 0 & \text{if } J_2 < 2s_1^2 \text{ or } \dot{W} < 0 \end{cases} \tag{10}$$

$J_2 \equiv \mathbf{S} \cdot \mathbf{S}$ and is self-consistently bounded by $2s_1^2$ (von Mises yield condition). The equation of motion is,

$$\rho \dot{\mathbf{v}} = -\nabla(p_{mat} + p_{mag}) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{\mu_0} + \nabla \cdot \mathbf{S} - \rho g \hat{\mathbf{y}}, \tag{11}$$

where p_{mat} and $p_{mag} \equiv B^2/2\mu_0$ are the isotropic material and magnetic pressures, respectively. The (incompressible) equation of state is

$$\nabla \cdot \mathbf{v} = \mathbf{0} \tag{12}$$

The magnetic diffusion equation is,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}, \tag{13}$$

where σ is electrical conductivity.

- The only governing equations different that the hydrodynamic case are Eq. 11 and Eq. 13
- In our case, $(\mathbf{B} \cdot \nabla)\mathbf{B} = 0$. This is satisfied in planar geometry (the limit of radius $r \gg h$ for shells of thickness h) if we constrain ourselves to a magnetic field of the form $\mathbf{B} = B(x, y)\hat{\mathbf{z}}$ resulting from a constant current per unit length in the z direction I which closes its circuit on the side that the plate is accelerating away from. To be consistent with this, only z invariant (two dimensional) perturbations are considered.
- Defining $p \equiv p_{mat} + p_{mag}$, then Eq. 10 becomes equivalent to the corresponding hydrodynamic Eq. 4.
- Furthermore, the boundary conditions expressed in terms of p are equivalent. In particular, $p \equiv p_0$ at the lower medium interface.
- Therefore, in the planar shear-free case, magnetic diffusion results in a 1:1 substitution of material pressure for isotropic magnetic pressure in an incompressible medium, provide the implied current has no effect on material properties. The result is equivalent evolution.
- What about Eq. 13? Though having no effect on the dynamics, once $\mathbf{v}$ is determined, it may be used to determine the partition between p_{mat} and p_{mag}.

III. ...with compressibility

- The above equivalence is traceable to the assumption that the relative drop in p_{mat} has no effect on local density.
- Compressibility introduces an equation of state involving p_{mat} independent of p_{mag}, requiring the coupling of the magnetic diffusion equation closure.
- As shown in Roderick 84, when significant compressibility of a magnetically accelerated medium *is* present (as for a plasma), magnetic diffusion has a significant effect on the dynamics, dissipating the streamers which appear in the nonlinear phase of the R-T instability by allowing them to broaden and merge.

[N. F. Roderick and T. W. Hussey, "A model for the saturation of the hydrodynamic Rayleigh-Taylor instability", J. Appl. Phys., **56**, 1387–1390, 1984]

- In Roderick 84 the interpretation was B-diffusion $\mapsto$ stabilization.
- A more applicable interpretation, in light of the above, is B-diffusion $\mapsto$ mass density redistribution $\mapsto$ stabilization.
- For p_0 =16 GPa, corresponding to 10 MA current imploding a cylinder of radius R = 1 cm or B=200 T, Al compresses by about 15%. These conditions are typical of the final stagnation conditions of a solid liner on Shiva Star or Pegasus. Relaxation to near standard density after B-field diffusion will clearly be insufficient to have a significant effect on streamer formation by this mechanism.

[J.M. Walsh "Equations of State of Metals from Shock Measurements", Phys. Rev. **97**, 1544-1556, 1955]

IV. ...with pressure dependent G and s_1

- One stabilization enhancement present with a gas driver that a diffused magnetic driver will *not* be able to take advantage of in the increase G and s_1 due to material pressure.
- In the model of Steinberg 96, G and s_1 are proportional, so there will be no change in the ordinate position on the stability diagram, which is dependent on the ratio of the two.

[D. J. Steinberg, "Equation of State and Strength Properties of Selected Materials (revised)", LLNL Tech Rep. UCRL-MA-106439 change 1, FEB 1996]

- In this model, the relative change in both terms due to compression and temperature change $\triangle T$ is

$$\frac{\triangle G}{G_0} = Ap_{mat}\left(\frac{\rho_0}{\rho}\right)^{\frac{1}{3}} - B\triangle T \tag{14}$$

  where, for Al (both 1100 and 6061-T6), $A = 6.52 \times 10^{-2}$ GPa^{-1} and $B = 1.16 \times 10^{-4}$ K^{-1}

- For p_0 =16 GPa (15% compression) there is less than 1 percent change in G. Clearly not an important effect.

V. ...with thermal softening and melting

- In regions where p is dominated by p_{mag}, Ohmic heating from magnetic diffusion will result in a temperature change of roughly $\triangle T \approx p_{mag}/C$, where C is heat capacity per unit volume ($C = 2.4 \times 10^6$ J/m^{3}K for Al)

[H. Knoepfel, *Pulsed High Magnetic Fields*, American Elsevier, New York, NY, 1970]

- For p_{mag} =1 GPa, corresponding to a 5 MA, current with $R = 2$ cm (50 T), $\triangle T \approx 416$ K and, from Eq. 14, G drops by 5%. These conditions are typical of the run-in phase of Shiva.

- A more significant effect will be melting. This occurs for $\triangle T = 920$ K, corresponding to a diffused $p_{mag} \approx 2$ GPa (70 T). Thermal softening is still only a 10% effect prior to meltins, but the molten layer is obviously R-T unstable.

VI. ...with m=0 MHD mode

- The external p_{mag} in an imploding liner actually falls off as $1/r^2$ due to ampere's law. This is a destabilizing effect relative to a gas driver.

- The condition for a gas driver model to be applicable is most easily expressed for short wavelengths. In this limit, the peak differential in $p_{mag} = p_0 R^2/r^2$ from the mean must smaller than the order of $\sqrt{3}s_1$ (uniaxial yield strength).

$$\frac{2p_0 q_{mc}}{R} \ll \sqrt{3}s_1 \qquad (15)$$

  where q_{mc} is the the maximum dynamic amplitude of a perturbation with initial amplitude arbitrarily close to, but less the critical amplitude for stability q_c.

- To use this, a plot similar the stability diagram may be made plotting q_{mc} instead of q_c.

- A simplier condition may be made for short wavelengths in rigid plastic limit where $q_{mc} \to q_c$ and,

$$q_c = \frac{4s_1 h}{\left(1 + e^{-2\pi h/\lambda}\right) p_0} \qquad (16)$$

[J. W. Miles, "Taylor Instability of a Flat Plate", General Dynamics Tech Rep. GAMD-7335]

- Discarding the exponent in Eq. 16 (short wavelengths) and combining with Eq. 15, this condition simplifies to

$$\frac{h}{R} \ll 0.2 \qquad (17)$$

- Typically, liners used are a few cm in initial radius, and 1 mm or so thick, so the condition is generally applicable during the initial run in phase.

VII. Conclusions

- The only apparent beneficial effect of magnetic drive in the geometry discussed results from the compressibility... and this is negligible for solids.
- Other effects considered are destabilizing relative to the gas driver case. However, very good agreement to gas driver models should be expected during the liner run in phase while $h/R \ll 0.2$ and for magnetic pressures up to 2 GPa (70 T) or so for Al.