Observable implications of equilibrium behavior on finite data

Susan K. Snyder *

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Abstract

We analyze the empirical implications of competitive equilibrium for finite data sets. We show that there are no restrictions on the model of competitive equilibrium when only aggregate-level data are observed, as long as there are at least as many consumers as goods or data points. In contrast, the restriction that the income distribution is unobserved but fixed will result in refutable conditions. In deriving these results we demonstrate a general framework for deriving testable restrictions on finite domains that exploits the semialgebraic structure of the problem.

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*email: Susan_Snyder@brown.edu. The paper greatly benefitted from the suggestions by an anonymous referee.

1 Introduction

The well-known theorems of Sonnenschein, Mantel, and Debreu can be seen to imply that the competitive equilibrium model is "irrefutable" — without strong assumptions on the form of utility functions, there is little we can say about the equilibrium prices and allocations that result (see Shafer and Sonnenschein (1982)). Work by Brown and Matzkin (1996) on the empirical implications of general equilibrium provided two important innovations in this field: they showed that competitive equilibrium does have refutable implications on the equilibrium manifold; and they explicitly incorporated the limitations of practical data sets into their restrictions by focusing on finite sets of observations of an economy.

These results raise two interrelated questions. First, how much data do we need in order to test the competitive equilibrium model? This question is of practical importance, relevant for cases when instead of data on prices and individual endowments, we have data on prices and aggregate endowments, or data disaggregated not by individual, but by group. Second, what are the key differences that result in Brown-Matzkin finding restrictions when Sonnenschein-Mantel-Debreu do not?

In this paper we address those questions by extending the Brown-Matzkin approach to derive testable restrictions of the competitive equilibrium model for different potential data sets. The semialgebraic nature of the problem allows us to set up a framework to derive testable restrictions over any set of data, or to incorporate additional assumptions into the model. We show that there are no restrictions on the model of competitive equilibrium when only aggregate-level data are observed, as long as there are at least as many consumers as goods or data points. Additionally, if we add the restriction that the income distribution is unobserved but fixed, as is traditional in much of the Sonnenschein-Mantel-Debreu literature, there are restrictions. It is necessary there be some variation in the endowments.

We choose to focus on the restrictions over a finite data set for a number of reasons. First, the assumption that what we observe is finite is realistic, so we can derive restrictions that are potentially empirically verifiable. Secondly, this assumption is useful because we can use semialgebraic methods to derive and analyze the testable restrictions of our models; questions of uniqueness, stability, and even general existence are not necessary precursors to deriving empirical implications. Third, this assumption leads to tests that are straightforward to implement, thus making a more direct connection between general equilibrium theory and actual tests of that theory. Few maintained hypotheses are necessary to put the tests into practice; in particular, the tests are nonparametric, so functional form assumptions are not required.

Chiappori et al. (2002) examine equilibrium manifold restrictions in a differentiable framework. The pattern of results obtained is similar to that obtained in the finite case. There are nonvacuous local restrictions on the equilibrium manifold when individual endowments are observed; further, that is enough data to completely identify the economy. With aggregate-level data alone, however, the restrictions vanish.

In the next section, we describe the basic framework of developing testable restrictions of competitive equilibrium behavior in a pure exchange model. Section 3 will present results in applying this framework under different sets of observables and assumptions. The conclusion follows.

2 The testable restrictions of the pure exchange model on the equilibrium manifold

In this section we provide some prerequisites on semialgebraic theory and make precise our model assumptions. We then describe Brown-Matzkin's results on the testable restrictions of pure exchange when prices and individual endowments are observed, and discuss how these testable restrictions can be readily generalized to cases where less data are observed.

2.1 Semialgebraic sets

A semialgebraic set is defined by a finite system of polynomial inequalities (these definitions and results are found in Mishra (1993)):

Definition 1 *A subset S of $\mathbf{R^m}$ is a* semialgebraic set *if it is the finite union of sets of the form:*

$$\{x \in \mathbf{R^m} : \quad f_i(x) = 0, i = 1, ..., k; \quad g_j(x) > 0, j = 1, ..., p\}$$

where f_i and g_j are polynomials with real coefficients.

Semialgebraic sets in $\mathbf{R^m}$ can also be defined in terms of propositional algebraic sentences, which are composed of: real constants, m real variables, arithmetic operators $(+\,,-\,,\cdot\,,/\,)$, binary relation operators $(=\,,\neq\,,>\,,<\,,\geq,\,,\leq\,)$, and Boolean connectives $(\neg\,,\Rightarrow\,,\wedge\,,\vee\,)$. If we add universal and existential quantifiers, $(\forall\,,\exists)$, ranging over the reals, we can compose *Tarski sentences*, which define subsets of $\mathbf{R^m}$ called *Tarski sets.*

For example, a Tarski set is defined by the quantified inequality:

$$\forall y \quad y^2 > x$$

The Tarski set will be composed of those values of x such that the statement is true. The set defined is semialgebraic, as it is the same set as defined by this inequality:

$$x < 0$$

The following theorem shows that given a Tarski set, we can always find an equivalent semialgebraic set, i.e., the quantifiers can be eliminated (see also Van Den Dries (1988)).

Tarski-Seidenberg Theorem *Every Tarski set is a semialgebraic set and the quantifier-free definition is derivable in finite time.*

In order to derive testable propositions over any particular data structure, the following corollary will be important:

Corollary 1 *Semialgebraic sets are closed under projection.*

To see this is true, consider a semialgebraic set described by $F(x) > 0$, $x = (x^1, x^2, ..., x^m) \in \mathbf{R^m}$. We can project this set to $\mathbf{R^k}$ for some $k < m$ by writing the system: $\exists(x^{k+1}, x^{k+2}, ..., x^m) \in \mathbf{R^{(m-k)}}$ s.t. $F(x^1, x^2, ..., x^k, x^{k+1}, ..., x^m) > 0$. By the Tarski-Seidenberg theorem this is a semialgebraic set over $\mathbf{R^k}$ and the equivalent quantifier-free definition, $G(x^1, x^2, ..., x^k) > 0$, is derivable in finite time.

2.2 Testable Propositions of Competitive Equilibrium

Consider a pure exchange economy with N agents each with preferences representable by a nonsatiated utility function U_n over ℓ commodities. Each agent has an initial endowment $\omega_n \in \mathbf{R}^{\ell}_{+}$.[1] We assume the aggregate endowment of each good will be strictly positive. A feasible allocation is a set of consumption vectors $\{x_n\}_{n=1}^{N}$, with $x_n \in \mathbf{R}^{\ell}_{+}$, such that $\sum_{n=1}^{N} x_n = \sum_{n=1}^{N} \omega_n$.[2] A competitive equilibrium is a feasible allocation $\{x_n\}_{n=1}^{N}$ and prices $p \in \mathbf{R}^{\ell}_{++}$ such that each x_n maximizes U_n subject to the budget constraint $p \cdot x_n \leq p \cdot \omega_n$.

We will parameterize the economy by the endowment vectors, $\{\omega_n\}$, while assuming agents' preferences do not change. For a given set of agents $\{U_n\}$, the *equilibrium correspondence* is defined by the set of equilibrium prices $W(\{\omega_n\})$ associated with each economy $\{\omega_n\}$. One could interpret this parameterization as an economy with changing endowments over time (where each agent has a one-period decisionmaking horizon), or as a cross-section of different groups of traders with the same utility functions.

The following theorem is an adaptation of Brown and Matzkin's Theorem 2, which describes the restrictions the competitive equilibrium model places on finite observations of prices and individual endowments (the equilibrium manifold).[3]

Theorem 1 (Brown-Matzkin) *Suppose $\langle p^r, \{\omega_n^r\}\rangle$ for $r = 1 \ldots R$ are given. Then there exists a set of continuous, strictly monotonic, strictly concave utility functions $\{U_n\}$ such that each p^r is an equilibrium price vector for the exchange economy $\langle \{U_n\}, \{\omega_n^r\}\rangle$ if and only if the following system $(*)$ is satisfied:*

$(*)$ *There exist vectors $\{x_n^r\}$ such that :*

Markets clear:

$$\sum_{n=1}^{N} x_n^r = \sum_{n=1}^{N} \omega_n^r \quad r = 1 \ldots R \tag{1.1}$$

[1] Where $\mathbf{R}^{\ell}_{+}$ represents the subset of $\mathbf{R}^{\ell}$ such that all elements are nonnegative, and $\mathbf{R}^{\ell}_{++}$ represents the subset of $\mathbf{R}^{\ell}$ such that all elements are strictly positive.

[2] For the rest of the paper, $\{x_n\}$ will be understood to mean the set $\{x_n\}_{n=1}^{N}$.

[3] In this statement of the theorem we have used the equivalence of a version of the Afriat inequalities that rationalize strictly concave utility functions and the Strong Axiom of Revealed Preference.

Budget constraints are satisfied:

$$p^r \cdot x_n^r = p^r \cdot \omega_n^r \quad r = 1 \dots R, \quad n = 1 \dots N \tag{1.2}$$

Individual consumptions are nonnegative:

$$x_n^r \geq 0 \quad r = 1 \dots R, \quad n = 1 \dots N \tag{1.3}$$

The Strong Axiom of Revealed Preference (SARP) is satisfied for all agents:

For every agent n, and for all subsets of size greater than 1 of R, $x_n^r \neq x_n^v$,

$$p^r \cdot x_n^r \geq p^r \cdot x_n^s \wedge p^s \cdot x_n^s \geq p^s \cdot x_n^t \wedge \dots \wedge p^u \cdot x_n^u \geq p^u \cdot x_n^v \Rightarrow p^v \cdot x_n^v < p^v \cdot x_n^r \tag{1.4}$$

Comment: The conditions can easily be rewritten for the case when individual incomes and the aggregate endowment are observed instead of individual endowments. Also note that these are the same restrictions we get on market demand when prices, aggregate consumption, and individual endowments or incomes are observed, though prices are endogenous in a competitive equilibrium model and exogenous in a market demand model.

The theorem states that equilibrium behavior will result in data (here, prices and individual endowments) described by the system $(*)$, which is a Tarski set. By the Tarski-Seidenberg theorem, this is a semialgebraic set in the price-individual endowment space, $(\mathbf{R}^\ell_{++} \times (\mathbf{R}^\ell_+)^N)^R$. The non-quantified description of the set can be derived in finite time; it would be in the form of a finite union of a finite set of polynomial inequalities defined over the data. Thus one can determine in finite time whether observations of prices and individual endowments are consistent with equilibrium, that is, whether the data are *rationalizable*. Moreover, these are all of the restrictions of the model; if the data are contained within this semialgebraic set, then there exist economies that could have generated the data.

This semialgebraic set is clearly nonempty for any number of consumers, goods, or observations: any example of utility functions and a series of endowment vectors such that equilibrium exists will result in prices, that, together with the endowments, are set members. Importantly, Brown and Matzkin found that the set is not the entire price-individual endowment space; there are observations of prices and individual endowments that are not consistent with $(*)$, and thus the model is potentially refutable on this data. Brown and Matzkin show this by example for two observations of a two-agent economy (with any finite number of goods); this can easily be extended to show the model is refutable for any number of agents for two or more observations.

The existence and characterization of testable restrictions depends on what kind of data can be observed. Because projections of a semialgebraic set are semialgebraic, Brown-Matzkin's theorem provides the basis for deriving testable restrictions over any subset of price-individual endowment data. If we observe less data than prices and individual endowments, we proceed by removing some variables from the "observable" set to the "unobservable" set, projecting to the new set of observables, and determining whether the resulting system is refutable.

3 The testable restrictions of the pure exchange model: general results

3.1 Aggregate data only

Consider the system $(*)$ in theorem 1 with the additional variables of aggregate endowments: $\omega^r \in \mathbf{R}^{\ell}_{++}$. These could be directly computed from the data: $\omega^r = \sum_{n=1}^{N} \omega_n^r \quad \forall r$. The system remains semialgebraic in prices and both individual and aggregate endowments. Suppose now we observe only aggregate endowments and prices. Because we do not observe individual endowments, they become *quantified* variables in $(*)$ — we can only ask whether there exist individual endowments consistent with equilibrium, given the other data we observe. We are projecting the restrictions of the equilibrium manifold, existing in the price-individual endowment-aggregate endowment space, to the price-aggregate endowment space.

Theorem 2 *Suppose $\langle p^r, \omega^r \rangle$ for $r = 1 \dots R$ are given. Choose an integer $N \geq 1$. Then there exists a set of continuous, strictly monotonic, strictly concave utility functions $\{U_n\}$*

such that each p^r is an equilibrium price vector for the exchange economy $\langle \{U_n\}, \{\omega_n^r\} \rangle$ if and only if the following system (**) is satisfied:

$(**)$ *There exist vectors $\{\omega_n^r\}, \{x_n^r\}$ such that:*

Conditions (1.1)-(1.4) are satisfied.

Individual endowments are nonnegative and sum:

$$\omega_n^r \geq 0 \quad r = 1 \ldots R \quad n = 1 \ldots N\,; \quad \sum_{n=1}^{N} \omega_n^r = \omega^r \quad r = 1 \ldots R \tag{2.5}$$

The set of $\langle p^r, \omega^r \rangle$ for $r = 1 \ldots R$ that satisfy $(**)$ is a semialgebraic set in $(\mathbf{R}_{++}^{\ell} \times \mathbf{R}_{++}^{\ell})^R$.

The proof follows immediately from corollary 1. Thus one can use these conditions to determine in finite time whether observations of prices and aggregate endowments are consistent with equilibrium. The next theorem states that the restrictions are vacuous, that is, the semialgebraic set described by $(**)$ is the entire space $(\mathbf{R}_{++}^{\ell} \times \mathbf{R}_{++}^{\ell})^R$, if we assume the number of agents is greater than or equal to either the number of goods or the number of observations.

Lemma 1 *Consider the set of individual endowment vectors for an agent: $\{\omega_n^1, \omega_n^2, ..., \omega_n^R\}$. Suppose this set is ordered for some arrangement of observations: $\omega_n^r \geq \omega_n^s \geq ... \geq \omega_n^t$, for some $r, s, ..., t$.[4] Then for any strictly positive prices, there exist a set of consumption vectors x_n^r such that the agent satisfies the budget constraint every period, consumption vectors are nonnegative, and SARP is satisfied.*

Proof: Consider the consumption vectors $x_n^r = \omega_n^r$, $r = 1, \ldots, R$. For any prices, $p^r \cdot x_n^r = p^r \cdot \omega_n^r$. For each set of observations of size greater than 1, without loss of generality label the observations from largest endowment to smallest endowment in order $r, s, t, ..., u, v$. For any strictly positive prices, $p^r \cdot x_n^r \geq p^r \cdot x_n^s$, $p^s \cdot x_n^s \geq p^s \cdot x_n^t$, ..., $p^u \cdot x_n^u \geq p^u \cdot x_n^v$. Also, $p^v \cdot x_n^r \geq p^v \cdot x_n^v$, but $p^v \cdot x_n^r = p^v \cdot x_n^v$ if and only if $x_n^r = x_n^v$. Thus SARP is satisfied.

[4] Where $\omega_n^r \geq \omega_n^s := \omega_n^r - \omega_n^s \in \mathbf{R}_{+}^{\ell}$.

Lemma 2 *Suppose* $\langle p^r, \omega^r \rangle$ *for* $r = 1 \ldots R$ *are given, and let* $N \geq R$. *Then there exists a set of nonnegative* $\{\omega_n^r\}$ *such that* $\sum_{n=1}^N \omega_n^r = \omega^r$, *and for each* n, *the set of endowments* $\{\omega_n^1, \omega_n^2, ..., \omega_n^R\}$ *is ordered for some arrangement of observations.*

Proof: Construct individual endowment vectors that sum to the aggregate endowment vectors in the following way: assign agent 1 all of ω^1, agent 2 all of ω^2, etc. After the R^{th} consumer, divide agent 1's endowment in 1/2 and assign agent $R+1$ the other half, etc., until all agents have been assigned. Thus each agent has a positive endowment in one period, and zero endowments in all the others.[5]

Lemma 3 *Suppose* $\langle p^r, \omega^r \rangle$ *for* $r = 1 \ldots R$ *are given, and let* $N \geq \ell$. *Then there exists a set of nonnegative* $\{\omega_n^r\}$ *such that* $\sum_{n=1}^N \omega_n^r = \omega^r$, *and for each* n, *the set of endowments* $\{\omega_n^1, \omega_n^2, ..., \omega_n^R\}$ *is ordered for some arrangement of observations.*

Proof: Construct individual endowment vectors that sum to the aggregate endowment vectors in the following way: assign agent 1 all of good 1 in every period, agent 2 all of good 2 in every period, etc. After the ℓ^{th} consumer, divide agent 1's endowment every period in 1/2 and assign agent $R+1$ the other half, etc., until all agents have been assigned. Thus each agent has a positive endowment of one good (the same good) every period and zero endowments of all other goods.[6]

Theorem 3 *Suppose* $\langle p^r, \omega^r \rangle$ *for* $r = 1 \ldots R$ *are given. Choose any integer* $N \geq \min(\ell, R)$. *Then there exists a set of continuous, strictly monotonic, strictly concave utility functions* $\{U_n\}$, *and a set of* $\{\omega_n^r\}$ *such that* $\sum_{n=1}^N \omega_n^r = \omega^r$, *and such that each* p^r *is an equilibrium price vector for the exchange economy* $\langle \{U_n\}, \{\omega_n^r\} \rangle$.

Corollary 2 *Any finite sequence of price-aggregate endowment data is consistent with Pareto-efficient behavior for some economy.*[7]

[5] Rather than requiring individual endowments be zero, we could choose some small endowment vector λ close enough to zero that the order is preserved and assign that vector in all periods but one for each consumer.

[6] Again we could identify some small $\lambda_k > 0$ fraction of each good k to assign rather than zero to make each agent's endowment of every good strictly positive.

[7] A less general version of this result appeared in an earlier unpublished paper.

Proof: When $N \geq \min(\ell, R)$, lemma 2 or 3 can be used to construct individual endowments that satisfy the conditions of lemma 1. The individual consumption vectors constructed in lemma 1 will be market clearing, and thus all the restrictions of theorem 2 are satisfied.[8] The corollary follows from the first welfare theorem.

Note that the above theorem also describes the restrictions on aggregate demand when prices and aggregate consumption are observed. Thus theorem 3 states that there are no restrictions on aggregate demand if we have no individual information, as long as the number of consumers is large enough. This is consistent with previous results which find no restrictions on the local behavior of market demand (see Sonnenschein (1973), Diewert (1977), Mantel (1977))[9] — their results were actually stronger as they also assumed restrictions on the income distribution.

Our results are instead on finite domains. Andreu (1982), once again assuming a restriction on the income distribution and assuming the number of consumers is greater than or equal to the number of goods, proved that there were no restrictions on market demand on finite domains, except for any restrictions that would derive from the nonnegativity of individual consumption. Theorem 3 shows that these nonnegativity conditions do not add any restrictions on aggregate data if the income distribution is unobserved or unrestricted.

If $N < \min(\ell, R)$, then the semialgebraic set described by $(**)$ can be strictly smaller than $(\mathbf{R}^{\ell}_{++} \times \mathbf{R}^{\ell}_{++})^R$. In these cases it may be possible to observe data that is not contained in the set and thus that cannot be rationalized as equilibrium behavior. Most of the previous results require at least ℓ consumers to decompose the market demand data; these results all presumed that the income distribution was fixed, however. Chiappori and Ekeland (1999) found that when the income distribution is unobserved and unrestricted, only $\ell/2$ consumers are required to locally decompose an analytic market demand function. We find that on a finite domain, even with no restrictions on the income distribution,

[8]Note we are constructing an autarkic equilibrium. This is not necessary; if we can find an allocation that satisfies the above conditions, use the prices to define budget sets, then pick any allocation on all consumers' budget sets to be the endowment.

[9]Assuming there are at least ℓ consumers, they find no restrictions, beyond homogeneity and the balance condition, on the value of the market demand function or the first derivatives; Chiappori and Ekeland (1999) show that there are no other restrictions if the market demand function is analytic.

ℓ consumers may still be required to rationalize the data. Suppose three of the R observations of a 3-good economy are: $p^1 = (10, 1, 1)$, $p^2 = (1, 10, 1)$, $p^3 = (1, 1, 10)$, $\omega^1 = (3, 1, 1)$, $\omega^2 = (1, 3, 1)$, and $\omega^3 = (1, 1, 3)$. If there are 3 or more consumers these 3 observations can be rationalized as described above, but if there are 2 consumers, there are no allocations that simultaneously satisfy SARP for each consumer (see Appendix). Thus, in small markets or economies (such as households) there is a greater likelihood of deriving empirically meaningful statements.[10]

When we look for restrictions on a finite domain, the number of distinct observations of data becomes relevant for determining whether restrictions are vacuous or not. Just as there are no restrictions imposed by utility maximization on one observation of price-consumption data, there are no restrictions imposed by competitive equilibrium on one observation of price-endowment data, no matter how many goods or consumers there are. Theorem 3 shows that we can rationalize market data with fewer than ℓ consumers if the number of observations of data is less than the number of goods. Thus for data sets for which the number of observations is less than the number of goods, and for which the number of consumers is small, there may be local restrictions, but no restrictions that can be detected with data.

Finally, note that we can find certain characteristics of data sets that will always lead to rationalizability, no matter what the number of consumers.

Theorem 4 *Consider the set of aggregate endowment vectors $\{\omega^1, \omega^2, ..., \omega^R\}$. Suppose this set is ordered for some arrangement of observations: $\omega^r \geq \omega^s \geq ... \geq \omega^t$, for some $r, s, ..., t$. Then for any strictly positive prices, and any finite positive integer N, there exists a set of continuous, strictly monotonic, strictly concave utility functions $\{U_n\}$, and a set of $\{\omega_n^r\}$ such that $\sum_{n=1}^N \omega_n^r = \omega^r$, and such that each p^r is an equilibrium price vector for the exchange economy $\langle\{U_n\}, \{\omega_n^r\}\rangle$.*

Proof: Construct individual endowment vectors that sum to the aggregate endowment vector by assigning each agent a $1/N$ fraction of the endowment vector each period.

[10] Having a small numbers of consumers in an aggregate demand model is not problematic; on the other hand, it does make the behavioral assumptions of competitive equilibrium more suspect.

Because the aggregate endowments were ordered, the individual endowments will be as well, and lemma 1 applies.

3.2 Assumptions on individual-level data

Consider the system $(*)$ in theorem 1 again with the additional variables of income share, $\delta_n^r \in \mathbf{R}\ \forall r\, \forall n$, where $p^r \omega_n^r = \delta_n^r p^r \omega^r$. As with aggregate endowments, these variables could be directly computed from the data and the system remains semialgebraic with their inclusion. Suppose now we observe only aggregate endowments and prices, but we add the restriction that income shares are fixed across time: $\delta_n^r = \delta_n\, \forall r$.

Theorem 5 *Suppose $\langle p^r, \omega^r \rangle$ for $r = 1 \ldots R$ are given. Then for any $N \geq 1$, there exists a set of continuous, strictly monotonic, strictly concave utility functions $\{U_n\}$, and a set of $\{\omega_n^r\} \in \mathbf{R}_+^\ell$ and $\{\delta_n\} \in \mathbf{R}_+$ such that $\sum \delta_n = 1$ and $p^r \cdot \omega_n^r = \delta_n\, p^r \cdot \omega^r$, and such that each p^r is an equilibrium price vector for the exchange economy $\langle \{U_n\}, \{\omega_n^r\} \rangle$ if and only if:*

$(***)$ *There exist vectors $\{\omega_n^r\}, \{x_n^r\}, \{\delta_n\}$ such that:*

Conditions (1.1)-(1.4), (2.5) are satisfied.

Individual income shares are fixed:

$$p^r \cdot \omega_n^r = \delta_n\, p^r \cdot \omega^r \quad r = 1 \ldots R \tag{5.6}$$

Individual income shares are nonnegative and sum:

$$\delta_n \geq 0 \quad n = 1 \ldots N\,;\ \sum_{n=1}^{N} \delta_n = 1 \tag{5.7}$$

The set of $\langle p^r, \omega^r \rangle$ for $r = 1 \ldots R$ that satisfy $(***)$ is a semialgebraic set in $(\mathbf{R}_{++}^\ell \times \mathbf{R}_{++}^\ell)^R$.

The proof again follows immediately from corollary 1. For a given N, the semialgebraic set described by $(***)$ can be strictly smaller than $(\mathbf{R}_{++}^\ell \times \mathbf{R}_{++}^\ell)^R$, and thus these restrictions can be nonvacuous on finite domains. For example, the data $p^1 = (1,3)$, $p^2 = (3,1)$, $\omega^1 = (1,7)$, $\omega^2 = (7,1)$ do not satisfy these conditions for $N = 2$ (see appendix). It is clear from the previous literature on market demand functions with fixed income distributions that there would have to be restrictions. Suppose that the $\{\omega^r\}$ were observations of market demand rather than endowments. Then the theorem could be interpreted to describe the restrictions on finite observations of market demand at prices $\{p^r\}$, assuming a fixed income distribution. Sonnenschein (1973) previously found that there were market demand functions with fixed income distributions that were not rationalizable for any number of consumers, and Shafer and Sonnenschein (1982) describe some general restrictions that result on finite domains.

Suppose further that aggregate endowments are fixed: $\omega^r = \hat{\omega}$. In this case, individual endowments can be assigned such that an agent's endowment does not change over time: $\omega_n^r = \hat{\omega_n}$. Then the restrictions of theorem 5 are vacuous (for any N), because agents can then consume the same bundle every period and SARP is trivially satisfied. This is the discrete analog to looking for empirical implications on a market excess demand function in equilibrium, or looking for empirical implications on the equilibrium manifold restricted to one set of endowments. There cannot be any restrictions because there is no variation in the exogenous variable, the endowment.

3.3 Incomplete individual-level data

Though the competitive equilibrium/aggregate demand model is generally vacuous on aggregate-level data, very little individual-level data is required to get restrictions: from Brown-Matzkin's results it follows that two observations of just one individual's income is enough to generate nonvacuous restrictions. In contrast, suppose instead of observing one agent's income, we observed income disaggregated by subsets of agents, with subsets of size greater than one. Then, once again, as long as the number of agents in each subset is at least as great as the number of goods or observations, there will be no restrictions. It is crucial to be able to pin down at least one agent's behavior, either through actually observing data unique to that one individual, or by being able to place tight bounds on a set of individuals' behavior when there is a small number of them relative to the number of goods or observations on the economy. This suggests it might be more empirically useful to have a data set containing incomplete individual information rather than a data set that contains greatly, but not completely, disaggregated data.

4 Conclusion

Brown-Matzkin's results are in stark contrast to the negative findings of Sonnenschein-Mantel-Debreu. There are two main sources of difference: Brown-Matzkin examine the case where one can observe income, and they limit their search for testable restrictions to finite domains. The results of this paper confirm that having observations or making assumptions on individual income is a key determinant of whether a model's testable restrictions will be nonvacuous. Analyzing the testable restrictions on a finite domain rather than concentrating on local conditions sometimes makes it more likely one will find nonvacuous results, as in the case where income distribution is unobserved but assumed to be fixed. On the other hand, because the number of observations available becomes important when looking for restrictions on a finite domain, sometimes we might have local restrictions that are not detectable with finite data.

Thus this gives guidelines as far as what data are needed to test models of competitive equilibrium and aggregate demand. In doing this we demonstrate a general framework for deriving testable restrictions that exploits the semialgebraic structure of the problem. The semialgebraic structure derives not from any restrictive assumptions about utility functions; it is a direct result of assuming that we only observe a finite set of data points. Since finiteness of data is a reasonable assumption, these methods should be useful as a guide in developing practical tests for general equilibrium models on different kinds of data sets, with different restricting hypotheses.

In order to apply these tests to data, some more guidelines for interpretation are in order. For example, if the testable restrictions are nonvacuous, yet the set of potential data that are not consistent with the restrictions is quite small, the likelihood of observing data that would reject the model is miniscule, whether the model is correct or not. Thus we would like to know not only if testable restrictions are nonvacuous, but also the likelihood of a false positive result. The results in this paper would be complemented by work in that direction.

5 Appendix: proofs

5.1

Here it is shown that the data $p^1 = (10, 1, 1), p^2 = (1, 10, 1), p^3 = (1, 1, 10), w^1 = (3, 1, 1), w^2 = (1, 3, 1), w^3 = (1, 1, 3)$ cannot be rationalized with only two consumers. Theorem 2 provides the conditions for these data to be rationalized (conditions (1.1)-(1.4) and (2.5)). Choose the number of consumers, N, to be 2. To simplify notation, let x^r, y^r be respectively consumer 1's and consumer 2's consumption in period r.

For each individual to satisfy SARP (condition 1.4), there must be no revealed preference cycle for any pair of observations or any ordered trio of observations (there are only two distinct ones). A consumption bundle x^1 is said to be revealed preferred to x^2 if $p^1x^1 \geq p^1x^2$. Thus for individual 1 to satisfy SARP, the following system must be satisfied:

$$\begin{aligned}
(p^1x^1 < p^1x^2 \quad \text{OR} \quad p^2x^2 < p^2x^1) \quad \text{AND} \\
(p^1x^1 < p^1x^3 \quad \text{OR} \quad p^3x^3 < p^3x^1) \quad \text{AND} \\
(p^2x^2 < p^2x^3 \quad \text{OR} \quad p^3x^3 < p^3x^2) \quad \text{AND} \\
(p^1x^1 < p^1x^2 \quad \text{OR} \quad p^2x^2 < p^2x^3 \quad \text{OR} \quad p^3x^3 < p^3x^1) \quad \text{AND} \\
(p^1x^1 < p^1x^3 \quad \text{OR} \quad p^3x^3 < p^3x^2 \quad \text{OR} \quad p^2x^2 < p^2x^1)
\end{aligned}$$

and similarly for consumer 2.

Claim 1: For this data, x^1 is not revealed preferred to (not RP) x^2 implies y^1 is revealed preferred to (RP) y^2. To prove, suppose not: x^1 not RP x^2, and y^1 not RP y^2. Then $p^1x^1 < p^1x^2$ and, using conditions (1.1) and (2.5) to substitute for y^r, $p^1(\omega^1 - x^1) < p^1(\omega^2 - x^2)$. Rewriting these conditions, they are $p^1(x^2 - x^1) > 0$ and $p^1(x^2 - x^1) < p^1(\omega^2 - \omega^1)$. Both conditions can be true only if $0 < p^1(\omega^2 - \omega^1)$. From our data, however, $p^1(\omega^2 - \omega^1) = -18$, thus there are no feasible consumption vectors such that both conditions are true.

Claim 2: For this data, x^1 not RP x^2 implies y^1 RP y^3. To prove, suppose not: x^1 not RP x^2, and y^1 not RP y^3. Then $p^1x^1 < p^1x^2$ and $p^1(\omega^1 - x^1) < p^1(\omega^3 - x^3)$, which can be rewritten $p^1x^1 > p^1(\omega^1 - \omega^3) + p^1x^3$, and with our data that becomes $p^1x^1 > 18+p^1x^3$. Both conditions can be true only if $18+p^1x^3 < p^1x^2$, or $18 < p^1(x^2-x^3)$. Consumption vectors are nonnegative and bounded above by the aggregate endowment. Thus the maximum possible value for $p^1(x^2 - x^3)$ is $p^1\omega^2$. With our data, this value is 14, and so there are no feasible consumption vectors such that both inequalities are satisfied.

Note: from the symmetry of the data, whenever x^r not RP x^s, y^r RP y^s, and y^r RP y^t, and similarly when the consumers are switched.

Suppose x^1 not RP x^2. Then from claim 1 y^1 RP y^2, and from claim 2 y^1 RP y^3. Then if consumer 2 is to satisfy SARP, it must be the case that y^2 not RP y^1, and y^3 not RP y^1. Again by the above conditions this implies that x^2 RP x^3, and x^3 RP x^2. Thus consumer 1 cannot satisfy SARP. The only other way for consumer 1 to satisfy SARP is if x^2 not RP x^1, which will lead to a similar contradiction. Therefore, both cannot simultaneously satisfy SARP with feasible, market-clearing consumption vectors.

5.2

Here it is shown that the data $p^1 = (1,3), p^2 = (3,1), w^1 = (1,7)$ and $w^2 = (7,1)$ do not satisfy the conditions of theorem 5 for $N = 2$.

With this data, claim 1 from the first proof holds again: if x^1 not RP x^2, then y^1 RP y^2.

Suppose x^1 not RP x^2, or $p^1x^1 < p^1x^2$. Let x_a^r, x_b^r, be consumer 1's consumption of the first good and the second good, respectively, in period r, and let δ be consumer 1's (fixed) income share, while consumer 2 receives share $1-\delta$. Applying the budget constraints and filling in the data, $x_b^2 = 22\delta - 3x_a^2$, so $p^1x^2 = 66\delta - 8x_a^2$ and x^1 not RP x^2 implies that $x_a^2 < (11/2)\,\delta$. Because the total endowment of good 2 in period 2 is 1, it must also be true that $22\delta - 3x_a^2 \leq 1$, so $x_a^2 \geq (22/3)\,\delta - 1/3$. For both of these conditions to be true, $(22/3)\,\delta - 1/3 < (11/2)\,\delta$, or $\delta < 2/11$.

Then from claim 1, if the second consumer is to satisfy SARP, it must be that y^2 not RP y^1, or $p^2y^2 < p^2y^1$. Applying the budget constraints and market clearing conditions, $x_a^1 = 22\,\delta - 3x_b^1$, so $p^2x^1 = 66\,\delta - 8x_b^1$, and since y^2 not RP y^1 means $p^2x^1 < 22\,\delta - 12$, then $x_b^1 > (11/2)\,\delta + 3/2$. It must also be true that x_a^1 is nonnegative, so $22\delta - 3x_b^1 \geq 0$, or $x_b^1 \leq (22/3)\,\delta$. For both of these conditions to be true, $(11/2)\,\delta + 3/2 < (22/3)\,\delta$, or $\delta > 9/11$. Thus we cannot pick income shares such that both consumer 1 and consumer 2 satisfy SARP in this way, and because of the symmetry of the data, we will have the same problem if we instead assume x^2 not RP x^1. There is no fixed income share and allocation that simultaneously satisfies SARP for each individual.

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