A PRETTY CURVE INTERPOLATING THE FIBONACCI SEQUENCE

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Abstract. We derive a smooth curve which oscillates within an envelope defined by scaled sinh and cosh functions. Its points of tangency are the Fibonacci numbers.

We will start with the most general case of a linear recursion and then specialize in stages to the Fibonacci sequence. Given any sequence $\{F_n\}$ satisfying a linear recursion $\sum_{i=0}^{k} a_i F_{n+i} = 0$, one can always substitute $F_n = \alpha^n$ for $\alpha \neq 0$ and obtain $\sum_{i=0}^{k} a_i \alpha^{n+i} = 0$, which is true iff $\sum_{i=0}^{k} a_i \alpha^i = 0$. This is simply a degree k polynomial (which we can assume is monic, so that $a_k = 1$), so there are k solutions in $\mathbb{C}$, namely $\{\alpha_1, \ldots, \alpha_k\}$, and if the solutions are distinct, any sequence satisfying the above recursion will be a linear combination, $F_n = \sum_{j=1}^{k} c_j \alpha_j^n$. Then, as with differential equations, one can match any given initial terms $F_0, \ldots, F_{k-1}$ by solving the linear problem $F_i = \sum_{j=1}^{k} c_j \alpha_j^i$, where $0 \leq i \leq k-1$, for the values of the c_j's.

One can fit a smooth curve to the sequence by setting $F(x) = \sum_{j=1}^{k} c_j \alpha_j^x$, although in general $F(x)$ will be complex. However, if all the coefficients in the recursion and all the initial data are real, then $F(x)$ will be real at each integer $x = n$, and so $\mathrm{Re}(F)$ gives a real curve through the points of the sequence.

In the special case $k = 2$, we have $\alpha^2 + a_1\alpha + a_0 = 0$, so there are two solutions, $\alpha_\pm = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$, and the general solution is the sum $F(x) = c_+\alpha_+^x + c_-\alpha_-^x$. If $a_0 < 0$, then both $\alpha_\pm$ are real and $\alpha_- < 0 < \alpha_+$, so we can write $\alpha_+^x = e^{x \ln \alpha_+}$ and $\alpha_-^x = (-1)^x(-\alpha_-)^x = e^{x\pi i} e^{x \ln(-\alpha_-)}$, so we get

$$\mathrm{Re}(F(x)) = c_+ e^{x \ln(\alpha_+)} + c_- \cos(x\pi) e^{x \ln(-\alpha_-)} \quad .$$

In the special case of the Fibonacci sequence, $a_0 = a_1 = 1$ and $F_0 = 0$, $F_1 = 1$, so $\alpha_\pm = \frac{1 \pm \sqrt{5}}{2}$, $c_\pm = \frac{\pm 1}{\sqrt{5}}$, and so $\mathrm{Re}(F(x)) = \frac{e^{x \ln(\phi)} - \cos(x\pi) e^{-x ln(\phi)}}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$. When x is an even integer, $\mathrm{Re}(F(x)) = \frac{\sqrt{5}}{2} \sinh(x \ln \phi)$, and when x is an odd integer, $\mathrm{Re}(F(x)) = \frac{\sqrt{5}}{2} \cosh(x \ln \phi)$. Therefore, $\mathrm{Re}(F(x))$ oscillates between the graphs of $\frac{\sqrt{5}}{2} \sinh(x \ln \phi)$ and $\frac{\sqrt{5}}{2} \cosh(x \ln \phi)$, giving the Fibonacci sequence as the points of tangency, which occur at $x = n$.

In fact, the oscillating part is $< \frac{1}{2}$ for $x > 1$, and so since F_n is an integer, it must be the nearest integer to $\frac{e^{n \ln(\phi)}}{\sqrt{5}} = \frac{\phi^n}{\sqrt{5}}$, for $n > 1$. This rule happens also to work for $n = 0$ and $n = 1$, so:

$$\text{for all } n > 0, F_n = \left\lfloor \frac{1}{2} + \frac{\phi^n}{\sqrt{5}} \right\rfloor \quad .$$

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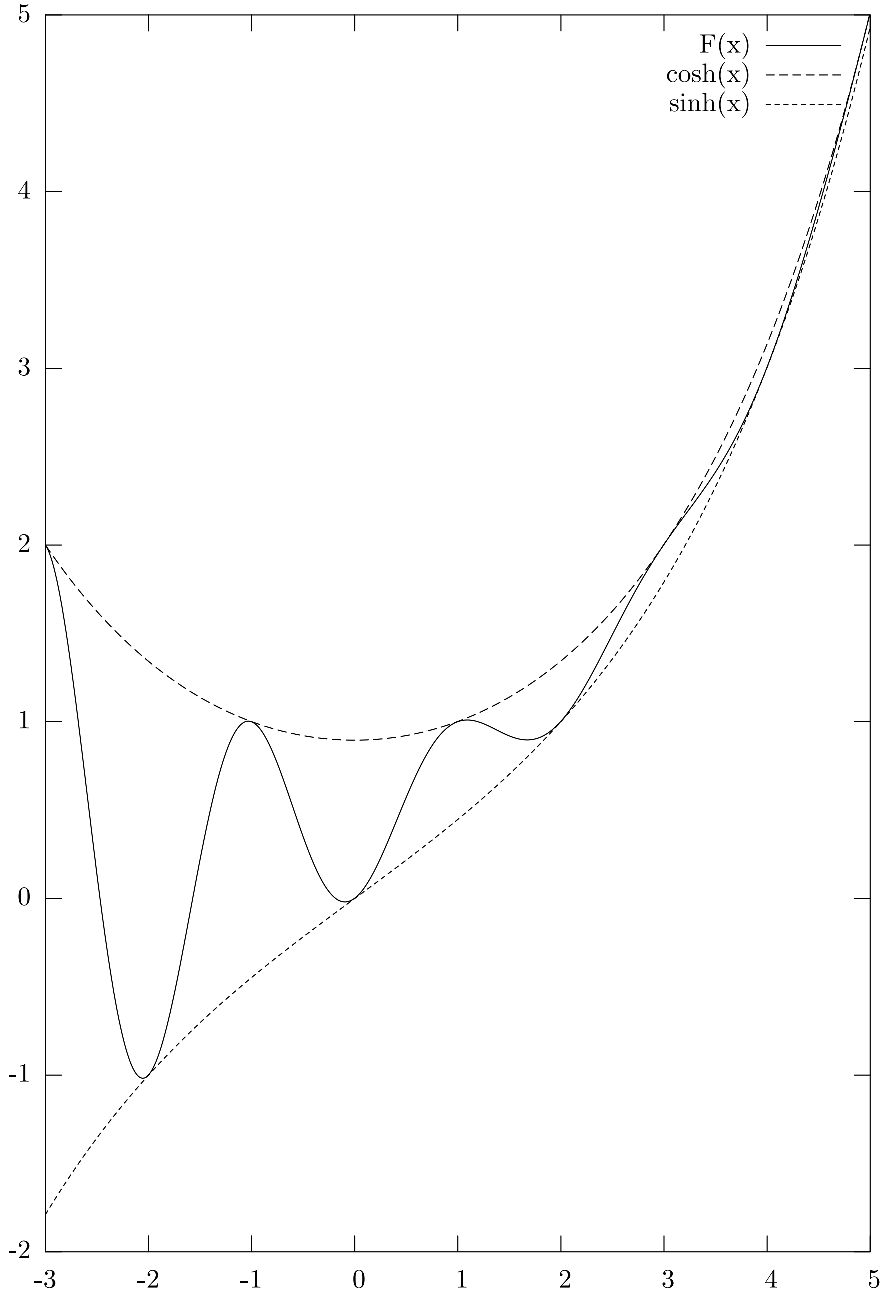

FIGURE 1. A smooth interpolation of the Fibonacci sequence.

Here is a graph illustrating the curve: