

Chapter 1 Dynamics of a Single Particle

Classical mechanics is the study of how matter moves in response to forces. Usually the forces are assumed to be given so that the origins of the forces are not considered. The basic physics that underlies classical mechanics is contained in Newton's laws of motion, and the subject may be seen as an exploration of the consequences of these laws. Despite their apparent simplicity, they have many non-trivial and surprising consequences. Until this century, they were believed to provide a precise description of dynamics at all scales from the astronomical to the microscopic. As a result, in the centuries following their formulation, many mathematical techniques were developed in order to allow them to be applied to systems of increasing complexity, such as ones with large numbers of mutually interacting particles, continuous media such as fluids and elastic solids and so on. One of their earliest applications was to the problem of planetary motion, which established their extraordinary accuracy. The fact that the existence and position of the planet Neptune were deduced by examining the irregularities in the orbits of the other planets and using perturbation theory demonstrates the sophistication to which methods of classical mechanics have been developed.

Classical mechanics also provides the background for studying many other areas of physics. The mathematical techniques developed for solving problems in classical mechanics have turned out to be useful in a variety of fields. Since classical mechanics is now thought to be an approximation to more complete descriptions such as relativistic and quantum mechanics, it is interesting to see in what ways these more accurate theories differ from classical mechanics and how these descriptions turn into the familiar classical description in the appropriate limits. For example, Newton's laws are no longer useful in quantum mechanics, and it is difficult to see how they can be generalised to this new situation. However, it turns out that the principle of least action, (which leads to the same predictions as Newton's laws in the classical regime) has a rather natural extension into quantum mechanics.

1.1 Newton's Laws for a point particle

For a single point particle of mass m , we may denote its position at time t by means of a position vector $\mathbf{r}(t)$ which specifies its displacement from some prescribed origin O . The velocity vector is defined to be

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}, \quad (1.1)$$

the momentum vector is $\mathbf{p}(t) = m\mathbf{v}(t)$, and the acceleration vector is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}. \quad (1.2)$$

For a single particle, only the first two of Newton's laws apply. If we are working in an inertial frame, the first law may be written

$$\mathbf{F}_{\text{tot}} = 0 \implies \mathbf{v} = \text{constant} \quad (1.3)$$

where \mathbf{F}_{tot} denotes the total force acting on the particle. The second law relates the total force to the rate of change of momentum,

$$\mathbf{F}_{\text{tot}} = \frac{d\mathbf{p}}{dt}. \quad (1.4)$$

If the mass is constant, we see that $\mathbf{F}_{\text{tot}} = m\mathbf{a}$. Calculation of the total force is done using **vector** addition.

These laws are all vector laws in three dimensional space, where each vector equality corresponds to equality of the corresponding components along any three linearly independent directions. We shall examine in more detail how to resolve vector equalities into their components later.

1.2 Motion in a Straight Line

When a particle is subject to forces which move it along a straight line, the vector equations reduce to scalar ones. We are usually given the force and require to solve for the motion. This requires the solution of ordinary differential equations, which can sometimes be done analytically. If no analytic solutions exist, numerical methods may be used. We shall discuss several analytic techniques which have been found to be useful for this problem.

1.2.1 Force specified as a function of time

If we are given the total force as a function of time $F(t)$, we can find the acceleration $a(t) = F(t)/m$. Integrating this twice yields the velocity $v(t)$ and the position $x(t)$ along the line:

$$v(t) = v_0 + \int_0^t a(\tau) d\tau \quad (1.5)$$

$$x(t) = x_0 + \int_0^t v(\tau) d\tau \quad (1.6)$$

where x_0 and v_0 are constants specifying the initial position and velocity of the particle. It is sometimes convenient to regard them as constants of integration. When the solution of a differential equation is expressed as explicit integrals, it is often said to be “reduced to quadratures”.

A specific example of a problem which falls into this class is when the force is constant. One then recovers the usual results for uniformly accelerated motion. Projectile motion in the absence of air friction is another example, since the motions in the horizontal and vertical directions are independent.

1.2.2 Force specified as a function of position

When a particle is attached by a spring to a fixed point, the force on the particle for small displacements from its equilibrium position is

$$F = -kx \quad (1.7)$$

where k is the spring constant. For a linear force law, the resulting differential equation

$$m \frac{d^2x}{dt^2} = -kx \quad (1.8)$$

is linear and may be solved by using the trial solution $x = e^{st}$. Substituting this trial solution yields the auxiliary equation

$$ms^2 = -k \quad (1.9)$$

which has solutions

$$s = \pm i \sqrt{\frac{k}{m}} \quad (1.10)$$

so that two linearly independent solutions of the differential equation are

$$\exp\left(i\sqrt{\frac{k}{m}}t\right) \text{ and } \exp\left(-i\sqrt{\frac{k}{m}}t\right). \quad (1.11)$$

The general solution is an arbitrary linear combination of this solution, which may be written in trigonometric form as

$$x(t) = A \cos\left(\sqrt{\frac{k}{m}}t\right) + B \sin\left(\sqrt{\frac{k}{m}}t\right) \quad (1.12)$$

for arbitrary constants A and B . This is the familiar solution for the simple harmonic oscillator. This method also applies to the case of damped simple harmonic motion where the force depends **linearly** both on velocity and position.

Often, the law of force $F(x)$ is a non-linear function of position, such as when it is due to gravity or electromagnetic interactions. In this case, the following method often works. Starting from Newton's second law,

$$m \frac{dv}{dt} = F(x) \quad (1.13)$$

This is complicated because three variables x , v and t are all present. It is possible to eliminate t by the substitution

$$\frac{dv}{dt} = \frac{dx}{dt} \frac{dv}{dx} = v \frac{dv}{dx} \quad (1.14)$$

so that we have

$$mv \frac{dv}{dx} = F(x) \quad (1.15)$$

This may be solved by separation of variables:

$$\int mv \, dv = \int F(x) \, dx \quad (1.16)$$

or

$$\frac{1}{2}mv^2 = \int F(x) \, dx + \text{constant} \quad (1.17)$$

This has the simple interpretation that the kinetic energy in a particular configuration is equal to the work done by the force on the particle and a constant (which is the kinetic energy in the original configuration). This is a simple example of the **work-energy theorem**.

Assuming that the integral can be carried out the result is to express v as some function of x . Let us write this as $v = g(x)$. Substituting $v = dx/dt$, we obtain

$$\frac{dx}{dt} = g(x) \quad (1.18)$$

which may be solved by separation of variables to give

$$\int \frac{dx}{g(x)} = t + \text{constant}. \quad (1.19)$$

As an example, we may solve the simple harmonic motion problem using this method. We find that

$$mv \frac{dv}{dx} = -kx \quad (1.20)$$

$$\frac{1}{2}mv^2 = -\int kx \, dx + \text{constant} = -\frac{1}{2}kx^2 + \text{constant} \quad (1.21)$$

If we suppose that $v = 0$ when $x = x_0$ so that x_0 is the amplitude of the oscillation, we can determine the constant. Solving for v yields

$$v = \sqrt{\frac{k}{m}(x_0^2 - x^2)} \quad (1.22)$$

If we plot a graph of v versus x , this is an ellipse. Substituting $v = dx/dt$ and separating the variables,

$$\int \frac{dx}{\sqrt{x_0^2 - x^2}} = \sqrt{\frac{k}{m}}t + \text{constant} \quad (1.23)$$

The integral on the left-hand side is $\sin^{-1}(x/x_0)$. If we write ϕ for the constant, we find

$$x = x_0 \sin \left(\sqrt{\frac{k}{m}}t + \phi \right) \quad (1.24)$$

which is the expected result.

1.2.3 Force specified as a function of velocity

This is useful for describing situations involving damping, where the frictional force can be expressed as a function of the speed of the particle. Depending on the range of velocities involved, the frictional force is approximately linearly or quadratically related to the speed. Care is often needed to get the direction of the force correct, especially if the direction of motion reverses. Another example of this class of problem is where an engine produces a constant power. Since power is equal to the product of force and velocity, a constant power corresponds to a force inversely proportional to the velocity.

Starting from Newton's second law,

$$m \frac{dv}{dt} = F(v) \quad (1.25)$$

we may separate the variables immediately to give

$$m \int \frac{dv}{F(v)} = t + \text{constant} \quad (1.26)$$

Alternatively, it may be convenient to write

$$mv \frac{dv}{dx} = F(v) \quad (1.27)$$

and to then separate the variables to give

$$m \int \frac{v}{F(v)} dv = x + \text{constant} \quad (1.28)$$

From these first integrals we get v as a function either of x or of t . A further separation of variables or integration yields x as a function of t .

1.3 Motion Along a Curve

Let us now consider a particle moving along some curve. We can label a point on the curve using a number s which is the arc length from some reference point on the curve. The equation of the curve may then be regarded as a function $\mathbf{r}(s)$ which gives the position vector of the point with label s . As the particle travels, we may denote its position on the curve by $s(t)$. Clearly the speed of the particle is $s'(t)$. The position vector of the particle is then the composition of the functions $\mathbf{r}(s)$ and $s(t)$, i.e., we may write (rather sloppily, from a mathematical viewpoint)

$$\mathbf{r}(t) = \mathbf{r}(s(t)) \quad (1.29)$$

We now wish to apply Newton's second law to the moving particle. In order to do this, we need to calculate both the velocity \mathbf{v} and the acceleration \mathbf{a} . The velocity is

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \frac{d\mathbf{r}}{ds} \quad (1.30)$$

Note that ds/dt is a scalar quantity which is the speed of the particle, while $d\mathbf{r}/ds$ is a vector. Since the speed is the magnitude of the velocity, this means that $d\mathbf{r}/ds$ must be a **unit vector**. The direction of the velocity is tangential to the curve at the point, and so we see that

$$\hat{\mathbf{u}} = \frac{d\mathbf{r}}{ds} \quad (1.31)$$

where $\hat{\mathbf{u}}$ is the unit tangent to the curve at the location of the particle. Thus

$$\mathbf{v} = v\hat{\mathbf{u}} \text{ where } v = \frac{ds}{dt}. \quad (1.32)$$

We note that in general, the unit vector $\hat{\mathbf{u}}$ changes as the particle moves along the curve.

Next, let us consider the acceleration of the particle. By definition, and using the product rule for differentiation,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\hat{\mathbf{u}}) = \frac{dv}{dt}\hat{\mathbf{u}} + v\frac{d\hat{\mathbf{u}}}{dt}. \quad (1.33)$$

Note that it is important to include the derivative of $\hat{\mathbf{u}}$, since the unit vector can change with time.

The first term is the acceleration **along** the curve, which is in the direction of the tangent vector. Not surprisingly, its magnitude is dv/dt or d^2s/dt^2 , which is the derivative of the speed of the particle.

The second term may be written as

$$v\frac{d\hat{\mathbf{u}}}{dt} = v\frac{ds}{dt}\frac{d\hat{\mathbf{u}}}{ds} = v^2\frac{d\hat{\mathbf{u}}}{ds} \quad (1.34)$$

If the curve is in fact a straight line, $\hat{\mathbf{u}}$ is unchanging along the curve and so $d\hat{\mathbf{u}}/ds$ is zero. The only acceleration we have to worry about is that along the line and the problem reduces to that considered above. On the other hand, if the curvature is non-zero, we have to find the meaning of $d\hat{\mathbf{u}}/ds$.

We notice that $d\hat{\mathbf{u}}/ds$ is a vector quantity. Its direction can be ascertained by considering the dot product with the unit tangent vector $\hat{\mathbf{u}}$. We see that (by the product rule, in reverse),

$$\hat{\mathbf{u}} \cdot \frac{d\hat{\mathbf{u}}}{ds} = \frac{1}{2} \frac{d}{ds} (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) = 0,$$

since $\hat{\mathbf{u}}$ is a unit vector and so $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}$ is always equal to one. If $d\hat{\mathbf{u}}/ds$ is non-zero, this can only mean that its direction is perpendicular to the direction of $\hat{\mathbf{u}}$.

We define the **unit normal** to the curve as being the unit vector in the direction of $d\hat{\mathbf{u}}/ds$. It is usually denoted by $\hat{\mathbf{n}}$. In order to find the magnitude of $d\hat{\mathbf{u}}/ds$, it is convenient to consider its value for a circle of radius R , since we expect it to depend upon the curvature of the curve at the point. A parametric equation for a circle of radius R centred at the origin is

$$\mathbf{r}(\theta) = \begin{pmatrix} R \cos \theta \\ R \sin \theta \end{pmatrix} \quad (1.35)$$

We now want to parameterise the equation by the arc length s . This is related to the angle θ by $s = R\theta$ and so

$$\mathbf{r}(s) = \begin{pmatrix} R \cos(s/R) \\ R \sin(s/R) \end{pmatrix} \quad (1.36)$$

The unit tangent vector is

$$\hat{\mathbf{u}} = \frac{d\mathbf{r}}{ds} = \begin{pmatrix} -\sin(s/R) \\ \cos(s/R) \end{pmatrix} \quad (1.37)$$

and so

$$\frac{d\hat{\mathbf{u}}}{ds} = -\frac{1}{R} \begin{pmatrix} \cos(s/R) \\ \sin(s/R) \end{pmatrix} \quad (1.38)$$

As expected, this is normal to the curve, and points towards the centre of the circle. The magnitude of $d\hat{\mathbf{u}}/ds$ is constant and equal to $1/R$, which is reasonable as the curvature of a circle is constant.

For a general curve, the normal vector $\hat{\mathbf{n}}$ points towards the instantaneous centre of curvature and

$$\frac{d\hat{\mathbf{u}}}{ds} = \frac{1}{R}\hat{\mathbf{n}} \quad (1.39)$$

where R is the instantaneous radius of curvature of the curve. Substituting this into equations (1.34) and (1.33) yields

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{u}} + \frac{v^2}{R}\hat{\mathbf{n}} \quad (1.40)$$

and we see that the acceleration along the normal is simply the centripetal acceleration required to keep the particle moving along the curve at speed v .

Notice that since the unit tangent $\hat{\mathbf{u}}$ and unit normal $\hat{\mathbf{n}}$ are at right angles to each other, we can consider an additional unit vector $\hat{\mathbf{b}}$ which completes the orthonormal set at the point.

$$\hat{\mathbf{b}} = \hat{\mathbf{u}} \times \hat{\mathbf{n}} \quad (1.41)$$

This is known as the **binormal vector**, and the set $\{\hat{\mathbf{u}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\}$ is called the **principal triad**.

Exercise: If the parametric representation of a plane curve is $(x(\tau), y(\tau))$ where τ is not necessarily the arc length, show that

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{x'^2 + y'^2}} \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad \left\| \frac{d\hat{\mathbf{u}}}{d\tau} \right\| = \frac{|x'y'' - x''y'|}{x'^2 + y'^2}, \quad \frac{ds}{d\tau} = \sqrt{x'^2 + y'^2} \quad \text{and that} \quad \frac{1}{R} = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}}. \quad (1.42)$$

Hence show that the radius of curvature of the parabola $y = x^2/(4a)$ of focal length a is

$$R = \frac{4a^2}{(4a^2 + x^2)^{3/2}} \quad (1.43)$$

so that at the vertex, the radius of curvature is twice the focal length.

1.4 Vector Laws and Coordinate Systems

Vectors are mathematical objects which represent physical quantities which have a **magnitude** and a **direction**. In order for two vectors to be equal, they must agree in both magnitude and direction. In order to calculate with vectors, we need to introduce coordinates so that the components of the vector with respect to the coordinate system can be found.

Perhaps the primary function of a coordinate system is to define the **position** of a point. Three commonly used coordinate systems are:

1. Cartesian coordinates (x, y, z) which measure the distances of a point from three fixed, mutually orthogonal axes,
2. Cylindrical polar coordinates (ρ, ϕ, z) where ρ and ϕ specify a position in the azimuthal plane and z is the height above this plane. They are related to Cartesian coordinates via

$$x = \rho \cos \phi, \quad y = \rho \sin \phi \quad \text{and} \quad z = z, \quad (1.44)$$

3. Spherical polar coordinates (r, θ, ϕ) where r is the distance from the origin, θ is the polar angle and ϕ is the azimuthal angle. They are related to Cartesian coordinates via

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi \quad \text{and} \quad z = r \cos \theta. \quad (1.45)$$

Once we have described how coordinates define positions, it is still necessary to describe how they define the components of a vector. In order to do this, we need to introduce **basis vectors** at each point. In principle, these can be any three linearly independent vectors, but we usually choose the basis so that it “respects the coordinate system”. Exactly what this means is open to some interpretation, and so different conventions exist.

1.4.1 Vectors in Cartesian Coordinates

For Cartesian coordinates, the basis vectors are always chosen to be the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ parallel to the coordinate axes. Given a vector \mathbf{v} , we can write

$$\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad (1.46)$$

The quantities v_x , v_y and v_z are three numbers known as the **components** of the vector \mathbf{v} in the Cartesian coordinate system. The column vector consisting of these three numbers will be denoted

$$\mathbf{v}^{\text{Cartesian}} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad (1.47)$$

so that

$$\mathbf{v} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \mathbf{v}^{\text{Cartesian}}. \quad (1.48)$$

Note that we should think of \mathbf{v} as being the “physical object” with magnitude and direction, while $\mathbf{v}^{\text{Cartesian}}$ is just a column of three numbers. Sometimes, we shall be sloppy and forget to make the distinction between \mathbf{v} and $\mathbf{v}^{\text{Cartesian}}$, but this can be a cause of some confusion unless one is careful. When one is thinking in terms of components forming a column vector, the row of basis vectors which premultiplies this to give the physical vector should always be kept in mind.

For the special case of Cartesian coordinates, the basis vectors are independent of position and the components of a vector are unchanged if the vector is “parallel transported” to another location. The position vector \mathbf{r} in Cartesian coordinates is $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.

1.4.2 Vectors in Other Coordinate Systems

For definiteness, let us consider the cylindrical polar coordinate system as our prototype. Given a point P with coordinates (ρ, ϕ, z) , we can define the basis vectors which respect the coordinate system as follows. The basis vector associated with a particular coordinate is found by incrementing that coordinate by an infinitesimal amount and fixing the other coordinates. The **direction** in which the point P moves is the direction of the basis vector. Consider for example the ϕ coordinate. If we increment only this coordinate by $\delta\phi$ and fix the other two, the change in the position vector of P is

$$\mathbf{r}(\rho, \phi + \delta\phi, z) - \mathbf{r}(\rho, \phi, z) \approx \frac{\partial \mathbf{r}}{\partial \phi} \delta\phi \quad (1.49)$$

We wish to define the basis vector associated with ϕ to be in this direction. In some subjects, notably relativity, it is usual to define

$$\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} \quad (1.50)$$

However, this is not of unit length, which leads to some complications. In our applications, we shall define the **unit vector** $\hat{\phi}$ as

$$\hat{\phi} = \frac{\partial \mathbf{r}}{\partial \phi} / \left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\| \quad (1.51)$$

We can write this in terms of the Cartesian basis vectors since

$$\mathbf{r} = \rho \cos \phi \hat{\mathbf{x}} + \rho \sin \phi \hat{\mathbf{y}} + z \hat{\mathbf{z}} \quad (1.52)$$

from which we see that

$$\frac{\partial \mathbf{r}}{\partial \phi} = -\rho \sin \phi \hat{\mathbf{x}} + \rho \cos \phi \hat{\mathbf{y}} \quad (1.53)$$

$$\left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\| = \sqrt{(-\rho \sin \phi)^2 + (\rho \cos \phi)^2} = \rho \quad (1.54)$$

and so

$$\hat{\phi} = \frac{\partial \mathbf{r}}{\partial \phi} / \left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\| = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}. \quad (1.55)$$

Similarly, it is easy to check that

$$\hat{\rho} = \frac{\partial \mathbf{r}}{\partial \rho} / \left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\| = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \quad (1.56)$$

$$\hat{\mathbf{z}} = \frac{\partial \mathbf{r}}{\partial z} / \left\| \frac{\partial \mathbf{r}}{\partial z} \right\| = \hat{\mathbf{z}} \quad (1.57)$$

Combining these into matrix form, we find

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.58)$$

The 3×3 matrix is the transformation matrix \mathbf{T} connecting the basis vectors in the two coordinate systems.

Given that we now have a transformation between the basis vectors, this induces a transformation between the components of vectors written in these two coordinate systems. For any vector \mathbf{v} ,

$$\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} = v_\rho \hat{\rho} + v_\phi \hat{\phi} + v_z \hat{\mathbf{z}} \quad (1.59)$$

$$= \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{\mathbf{z}} \end{pmatrix} \begin{pmatrix} v_\rho \\ v_\phi \\ v_z \end{pmatrix} \quad (1.60)$$

$$= \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \mathbf{v}^{\text{Cartesian}} = \begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{\mathbf{z}} \end{pmatrix} \mathbf{v}^{\text{Cylindrical Polars}} \quad (1.61)$$

By the rule (1.58) for the transformation of the basis vectors, we find

$$\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_\rho \\ v_\phi \\ v_z \end{pmatrix}. \quad (1.62)$$

Since the basis vectors are linearly independent,

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_\rho \\ v_\phi \\ v_z \end{pmatrix} \quad (1.63)$$

or

$$\mathbf{v}^{\text{Cartesian}} = \mathbf{T} \mathbf{v}^{\text{Cylindrical Polars}} \quad (1.64)$$

Which shows how the components of a vector with respect to the two coordinate systems transform.

Note that:

- The basis vectors $\hat{\rho}, \hat{\phi}$ are different at each point. In general, all three basis vectors can be position dependent.
- When a vector is parallel transported, its components with respect to the cylindrical polar coordinate system change.
- In the case of cylindrical polars, the basis vectors $\hat{\rho}, \hat{\phi}$ and $\hat{\mathbf{z}}$ form an **orthonormal basis**. This is reflected in the fact that the transformation matrix \mathbf{T} is **orthogonal** so that $\mathbf{T}^t = \mathbf{T}^{-1}$. This means that dot products may be readily computed,

$$\mathbf{u} \cdot \mathbf{v} = (u_\rho \hat{\rho} + u_\phi \hat{\phi} + u_z \hat{\mathbf{z}}) \cdot (v_\rho \hat{\rho} + v_\phi \hat{\phi} + v_z \hat{\mathbf{z}}) \quad (1.65)$$

$$= u_\rho v_\rho (\hat{\rho} \cdot \hat{\rho}) + u_\rho v_\phi (\hat{\rho} \cdot \hat{\phi}) + \dots + u_z v_z (\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \quad (1.66)$$

$$= u_\rho v_\rho + u_\phi v_\phi + u_z v_z = (\mathbf{u}^{\text{Cylindrical Polars}})^t (\mathbf{v}^{\text{Cylindrical Polars}}) \quad (1.67)$$

If we did not normalise our basis vectors, the expression for $\mathbf{u} \cdot \mathbf{v}$ would have been more complicated, and we would have had to introduce a metric tensor explicitly.

1.5 Velocity and Acceleration in General Coordinate Systems

Given the position vector \mathbf{r} , the velocity and acceleration are defined as $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$ respectively. We would like to see how these appear in terms of the components of these vectors. For Cartesian coordinates, this is trivial:

$$\mathbf{r} = x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \mathbf{r}^{\text{Cartesian}} \quad (1.68)$$

$$\mathbf{v} = \dot{x}\hat{\mathbf{x}}+\dot{y}\hat{\mathbf{y}}+\dot{z}\hat{\mathbf{z}} = \begin{pmatrix} \dot{\hat{x}} & \dot{\hat{y}} & \dot{\hat{z}} \end{pmatrix} \mathbf{v}^{\text{Cartesian}} \quad (1.69)$$

where the dots indicate derivatives with respect to time. Thus we see that

$$\mathbf{v}^{\text{Cartesian}} = \dot{\mathbf{r}}^{\text{Cartesian}} \quad (1.70)$$

Similarly,

$$\mathbf{a} = \ddot{x}\hat{\mathbf{x}}+\ddot{y}\hat{\mathbf{y}}+\ddot{z}\hat{\mathbf{z}} = \begin{pmatrix} \ddot{\hat{x}} & \ddot{\hat{y}} & \ddot{\hat{z}} \end{pmatrix} \mathbf{a}^{\text{Cartesian}} \quad (1.71)$$

$$\mathbf{a}^{\text{Cartesian}} = \dot{\mathbf{v}}^{\text{Cartesian}} \quad (1.72)$$

These were easy because the basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are independent of time.

For cylindrical polar coordinates, on the other hand, the position vector is

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}}+z\hat{\mathbf{z}} = \begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \end{pmatrix} \mathbf{r}^{\text{Cylindrical Polars}} \quad (1.73)$$

To calculate \mathbf{v} , we have to realise that the unit vectors can be time-dependent. Thus we need the time-derivatives of the unit vectors. For example, since

$$\hat{\boldsymbol{\rho}} = \cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}} \quad (1.74)$$

$$\frac{d\hat{\boldsymbol{\rho}}}{dt} = -\sin\phi\dot{\phi}\hat{\mathbf{x}}+\cos\phi\dot{\phi}\hat{\mathbf{y}} = \dot{\phi}\hat{\boldsymbol{\phi}} \quad (1.75)$$

Similarly,

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\boldsymbol{\rho}} \text{ and } \frac{d\hat{\mathbf{z}}}{dt} = 0 \quad (1.76)$$

So when we differentiate \mathbf{r} , we obtain

$$\frac{d\mathbf{r}}{dt} = \dot{\rho}\hat{\boldsymbol{\rho}}+\rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}}+z\frac{d\hat{\mathbf{z}}}{dt} \quad (1.77)$$

$$= \dot{\rho}\hat{\boldsymbol{\rho}}+\rho\dot{\phi}\hat{\boldsymbol{\phi}}+\dot{z}\hat{\mathbf{z}} \quad (1.78)$$

By definition this is $\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \end{pmatrix} \mathbf{v}^{\text{Cylindrical Polars}}$. Thus we have that

$$\mathbf{r}^{\text{Cylindrical Polars}} = \begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix} \text{ and } \mathbf{v}^{\text{Cylindrical Polars}} = \begin{pmatrix} \dot{\rho} \\ \rho\dot{\phi} \\ \dot{z} \end{pmatrix} \quad (1.79)$$

and so $\mathbf{v}^{\text{Cylindrical Polars}} \neq \dot{\mathbf{r}}^{\text{Cylindrical Polars}}$, even though $\mathbf{v} = \dot{\mathbf{r}}$ when regarded as vectors.

Differentiating once again,

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\rho}\hat{\boldsymbol{\rho}}+\dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}}+\rho\ddot{\phi}\hat{\boldsymbol{\phi}}+\rho\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \ddot{z}\hat{\mathbf{z}} + \dot{z}\frac{d\hat{\mathbf{z}}}{dt} \quad (1.80)$$

$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}} \quad (1.81)$$

and so

$$\mathbf{a}^{\text{Cylindrical Polars}} = \begin{pmatrix} \ddot{\rho} - \rho\dot{\phi}^2 \\ \rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} \\ \ddot{z} \end{pmatrix} \quad (1.82)$$

Exercise: Show that in the spherical polar coordinate system, with basis vectors $(\hat{\mathbf{r}} \ \hat{\boldsymbol{\theta}} \ \hat{\boldsymbol{\phi}})$,

$$\mathbf{r}^{\text{Spherical Polars}} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}^{\text{Spherical Polars}} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r \sin \theta \dot{\phi} \end{pmatrix} \quad (1.83)$$

$$\mathbf{a}^{\text{Spherical Polars}} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r \sin \theta \cos \theta \dot{\phi}^2 \\ r \sin \theta \ddot{\phi} + 2\dot{r} \sin \theta \dot{\phi} + 2r\dot{\theta} \cos \theta \dot{\phi} \end{pmatrix} \quad (1.84)$$

1.6 Conservation Laws

These are consequences of the laws of motion. For a single particle, they are fairly trivial, but their importance arises from their application to multiparticle systems for which the dynamics may be too complicated to analyse in detail. Using a conservation law, we can know that certain quantities remain constant in time even though these quantities are (in general) functions of all the particles whose individual motions we are not trying to trace out.

1.6.1 Linear Momentum

If **no force** acts on the particle $\mathbf{F} = \mathbf{0}$, and then by the second law, $\mathbf{p} = \text{constant}$. Note that \mathbf{F} refers to the **total** force acting on the particle.

1.6.2 Angular Momentum

The angular momentum of a particle is always defined with respect to some origin O . If \mathbf{r} is the position vector of the particle with respect to this origin, the angular momentum \mathbf{L} is defined to be

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad (1.85)$$

where \mathbf{p} is the linear momentum of the particle. Recall that the vector cross product is defined such that its length is

$$\|\mathbf{L}\| = \|\mathbf{r}\| \|\mathbf{p}\| \sin \theta \quad (1.86)$$

where θ is the angle between \mathbf{r} and \mathbf{p} and the direction of \mathbf{L} is such that the triple $(\mathbf{r}, \mathbf{p}, \mathbf{L})$ form a right-handed triad (i.e., the right-hand rule). In component form

$$\mathbf{L} = (yp_z - zp_y) \hat{\mathbf{x}} + (zp_x - xp_z) \hat{\mathbf{y}} + (xp_y - yp_x) \hat{\mathbf{z}} \quad (1.87)$$

If a force \mathbf{F} acts on the particle, the **torque** \mathbf{N} that the force exerts on the particle with respect to O is defined to be

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} \quad (1.88)$$

We see that

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) = \left(\frac{d\mathbf{r}}{dt} \right) \times \mathbf{p} + \mathbf{r} \times \left(\frac{d\mathbf{p}}{dt} \right) \\ &= \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} \\ &= \mathbf{N} \end{aligned} \quad (1.89)$$

since \mathbf{p} is parallel to \mathbf{v} and so $\mathbf{v} \times \mathbf{p} = \mathbf{0}$. Thus if the applied torque about O is equal to zero, the angular momentum remains constant.

1.6.3 Energy

If a particle is acted upon by a force $\mathbf{F}(t)$ during an interval of time $[t_1, t_2]$ during which it moves along the path $\mathbf{r}(t)$, the **work** done by the force is defined to be

$$W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \quad (1.90)$$

where $\mathbf{r}_1 = \mathbf{r}(t_1)$ and $\mathbf{r}_2 = \mathbf{r}(t_2)$ and this path integral is a shorthand for

$$W_{12} = \int_{t_1}^{t_2} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} dt \quad (1.91)$$

Using Newton's second law and the definition of velocity, we see that

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = \frac{1}{2} \int_{t_1}^{t_2} m \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) dt \\ &= \left[\frac{1}{2} m v^2 \right]_{t_1}^{t_2} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \end{aligned} \quad (1.92)$$

where $v^2 \equiv \mathbf{v} \cdot \mathbf{v}$, $\mathbf{v}_1 = \mathbf{v}(t_1)$ and $\mathbf{v}_2 = \mathbf{v}(t_2)$.

Identifying $T = \frac{1}{2} m v^2$ as the kinetic energy of the particle, we have derived the **work-energy theorem** which asserts that the change in **kinetic energy** is equal to the work done by the total (resultant) force on the particle.

$$T_2 - T_1 = W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \quad (1.93)$$

In the work-energy theorem, we do not make any statement about the source of the force \mathbf{F} which acts upon the particle. Depending on the nature of this force, it is sometimes possible to say more. An important class of forces are those which are **conservative**. The force (and the system) is said to be **conservative** if:

1. The force is a function of the **position** alone (i.e., once we know the particle is at a particular \mathbf{r} , the force is known and does not depend on the time at which we reach that position).
2. The work done by the force depends only upon the starting and ending positions of the particle and **not** upon the path taken between these positions. In particular, if the starting and ending positions coincide, the work done must be zero. A necessary and sufficient condition for a force to be conservative is that

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad (1.94)$$

where the circle around the integral sign indicates that the integral is carried out for **any** closed path.

By Stokes theorem, the line integral over a closed path may be replaced by one over an area whose boundary is the path:

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \quad (1.95)$$

For a conservative force, this integral is zero for every closed curve and so we require that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point. If this is the case, there exists a scalar-valued function $V(\mathbf{r})$ such that

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \quad (1.96)$$

The work done by a conservative force in taking a particle from \mathbf{r}_1 to \mathbf{r}_2 is thus given by

$$\begin{aligned} W_{12} &= - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla V(\mathbf{r}) \cdot d\mathbf{r} \\ &= - \int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} + \frac{\partial V}{\partial z} \frac{dz}{dt} \right) dt \\ &= V(\mathbf{r}(t_1)) - V(\mathbf{r}(t_2)) \equiv V_1 - V_2 \end{aligned} \quad (1.97)$$

This depends only on the difference of the function V at the endpoints, and so is evidently path-independent. Combining equations (1.93) and (1.97) we see that

$$T_2 - T_1 = W_{12} = V_1 - V_2 \quad (1.98)$$

or

$$T_1 + V_1 = T_2 + V_2. \quad (1.99)$$

It thus makes sense to call $V(\mathbf{r})$ the **potential energy**. For a conservative system, the total energy, which is the **sum** of the kinetic and potential energies, is conserved.

1.7 The Central Force Problem

Here we consider the motion of a single particle which is acted upon by a force which is directed towards the origin. Such a force is said to be a **central force**. The magnitude of the force depends only on the distance between the particle and the origin. This is an approximate description of the motion of a planet around the sun, where the sun is assumed to be so massive that the centre of mass of the system is assumed to coincide with the position of the sun

Let us choose the axes so that the initial velocity of the particle is in the xy plane. Since the force always acts towards the origin, this force also lies in the xy plane. This means that the particle will always remain in the plane throughout the motion. We can thus use plane polar coordinates (ρ, ϕ) to describe the motion. The assumption of a central force means that the force vector is directed along $\hat{\rho}$, and has no component along $\hat{\phi}$, i.e., we can write

$$\mathbf{F} = f(\rho) \hat{\rho} \quad (1.100)$$

In polar coordinates, the acceleration is

$$\mathbf{a} = (\ddot{\rho} - \rho\dot{\phi}^2) \hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \hat{\phi} \quad (1.101)$$

and so, equating components, we find that

$$m(\ddot{\rho} - \rho\dot{\phi}^2) = f(\rho) \quad (1.102)$$

$$\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = 0 \quad (1.103)$$

Note that

$$\frac{d}{dt}(\rho^2\dot{\phi}) = 2\rho\dot{\rho}\dot{\phi} + \rho^2\ddot{\phi} = \rho(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \quad (1.104)$$

and so, by equation (1.103) $\rho^2\dot{\phi}$ is a constant of the motion. This is just the conserved angular momentum of the particle, since

$$\mathbf{l} = m(\mathbf{r} \times \mathbf{v}) = m(\rho\hat{\rho} \times (\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi})) = m\rho^2\dot{\phi}\hat{\mathbf{z}} \quad (1.105)$$

So, if we denote $\|\mathbf{l}\|$ by l , we find

$$\rho^2\dot{\phi} = \frac{l}{m} \quad (1.106)$$

The rate at which area is swept out by the particle is

$$\frac{dA}{dt} = \frac{1}{2}\rho^2 \left(\frac{d\phi}{dt} \right) = \frac{l}{2m} \quad (1.107)$$

which is constant. This is Kepler's second law which is true for *any* central force. Conservation of angular momentum about a point may be regarded as a test for whether the force acting on the particle always passes through that point.

1.7.1 Equation for the Orbit

The equation for the orbit is an expression for ρ as a function of ϕ . In order to obtain this, we need to eliminate the time between the equations (1.102) and (1.106). Using (1.106), we have

$$\dot{\phi} = \frac{l}{m\rho^2} \quad (1.108)$$

and so

$$\frac{d\rho}{dt} = \frac{d\rho}{d\phi} \frac{d\phi}{dt} = \frac{l}{m\rho^2} \left(\frac{d\rho}{d\phi} \right) \quad (1.109)$$

Similarly,

$$\frac{d^2\rho}{dt^2} = \frac{l}{m\rho^2} \frac{d}{d\phi} \left(\frac{d\rho}{dt} \right) = \frac{l}{m\rho^2} \frac{d}{d\phi} \left(\frac{l}{m\rho^2} \frac{d\rho}{d\phi} \right) \quad (1.110)$$

so that (1.102) becomes

$$\frac{l^2}{m} \left\{ \frac{1}{\rho^2} \frac{d}{d\phi} \left(\frac{1}{\rho^2} \frac{d\rho}{d\phi} \right) - \frac{1}{\rho^3} \right\} = f(\rho) \quad (1.111)$$

The derivative may be simplified considerably by introducing $u = \rho^{-1}$ since

$$\frac{d\rho}{d\phi} = -\frac{1}{u^2} \frac{du}{d\phi} \text{ so that } \frac{1}{\rho^2} \frac{d\rho}{d\phi} = -\frac{du}{d\phi} \quad (1.112)$$

Thus

$$-\frac{l^2 u^2}{m} \left(\frac{d^2 u}{d\phi^2} + u \right) = f \quad (1.113)$$

We shall use this to find the orbit, given f . It may also be used to find f if the form of the orbit is known.

For the inverse square law of force,

$$f = -\frac{k}{\rho^2} = -ku^2 \quad (1.114)$$

where $k > 0$ for an attractive force. The equation of the orbit (1.113) then reduces to

$$\frac{d^2 u}{d\phi^2} + u = \frac{mk}{l^2} \quad (1.115)$$

which may readily be solved to yield

$$u = \frac{mk}{l^2} + A \cos(\phi - \phi_0) \quad (1.116)$$

where A and ϕ_0 are constants of integration. A is related to the energy of the particle and ϕ_0 is related to the orientation of the axis of the orbit. In terms of ρ , we have

$$\frac{l^2/(mk)}{\rho} = 1 + \frac{Al^2}{mk} \cos(\phi - \phi_0) \quad (1.117)$$

This is a conic section with eccentricity $Al^2/(mk)$ and semi-latus rectum l^2/mk . The centre of the force coincides with the focus.

1.7.2 Energy Considerations

An inverse square force law $\mathbf{F} = -k\hat{\mathbf{r}}/r^2$ can be derived from a potential energy function $V(r) = -k/r$. If we now restrict ourselves to the xy plane and use plane polar coordinates (ρ, ϕ) the total energy of a particle in orbit is

$$E = T + V = \frac{1}{2}m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 \right) - \frac{k}{\rho} \quad (1.118)$$

By the conservation of angular momentum, $\dot{\phi} = l/(m\rho^2)$ and so

$$E = \frac{1}{2}m\dot{\rho}^2 + \left(\frac{l^2}{2m\rho^2} - \frac{k}{\rho} \right) \quad (1.119)$$

We want to find E in terms of the orbit, and so we again try to eliminate t using (1.109). This gives

$$E = \frac{l^2}{2m} \left[\left(\frac{1}{\rho^2} \frac{d\rho}{d\phi} \right)^2 + \frac{1}{\rho^2} \right] - \frac{k}{\rho} \quad (1.120)$$

which simplifies on setting $u = \rho^{-1}$ to

$$E = \frac{l^2}{2m} \left[\left(\frac{du}{d\phi} \right)^2 + u^2 \right] - ku. \quad (1.121)$$

We wish to relate A in the equation (1.116) for the orbit to the energy. Substituting it into the expression for E we find

$$E = \frac{A^2 l^2}{2m} - \frac{mk^2}{2l^2} \quad (1.122)$$

or

$$A = \sqrt{\frac{2mE}{l^2} + \left(\frac{mk}{l^2} \right)^2} \quad (1.123)$$

The eccentricity of the orbit is

$$e = \frac{Al^2}{mk} = \sqrt{1 + \frac{2l^2 E}{mk^2}} \quad (1.124)$$

1. If $E > 0$, then $e > 1$ and the orbit is a hyperbola,
2. If $E = 0$, then $e = 1$ and the orbit is a parabola,
3. If $-mk^2/(2l^2) < E < 0$, then $0 < e < 1$ and the orbit is an ellipse,
4. If $E = -mk^2/(2l^2)$, then $e = 0$ and the orbit is a circle.

For the circular orbit, $\rho^{-1} = mk/l^2$ or $\rho = l^2/mk$ as may be checked by equating the centripetal force to the gravitational attraction.

1.7.3 The Effective Potential and the Centrifugal Barrier

The total energy is

$$E = \frac{1}{2} m \dot{\rho}^2 + \left(\frac{l^2}{2m\rho^2} - \frac{k}{\rho} \right) \quad (1.125)$$

where we have eliminated ϕ using the conservation of angular momentum. We may regard this as a one-dimensional problem for ρ with an “effective potential energy”

$$V_{\text{eff}} = \frac{l^2}{2m\rho^2} - \frac{k}{\rho} \quad (1.126)$$

If we have a particle moving in one dimension (along the ρ direction) under the influence of this potential energy function, its position $\rho(t)$ is identical with the radial coordinate of the actual particle in orbit. The term $l^2/(2m\rho^2)$ is called a “centrifugal barrier” since it effectively gives a repulsive force which prevents the particle from falling into the centre of force. Physically, this is because the conservation of angular momentum causes the particle to speed up as it gets closer to the centre.

For a given value of the total energy E , the intersections $V_{\text{eff}} = E$ give the boundaries of the orbit.

1.7.4 Time in Orbit

Using Kepler's second law, the time to travel from one point in the orbit to another is $2mA/l$ where A is the area swept out by a line joining the particle to the force centre as the particle moves between these points. The time for a complete orbit (the period) is thus given by

$$T = \frac{2m}{l} (\text{area of orbit}) \quad (1.127)$$

For an ellipse of semimajor axis a and semiminor axis b , the area is given by πab , where

$$a = \frac{l^2}{mk(1-e^2)} \text{ and } b = \frac{l^2}{mk\sqrt{1-e^2}}. \quad (1.128)$$

So we see from $T = 2m\pi ab/l$ that

$$\frac{a^3}{T^2} = \frac{k}{4\pi^2 m} \quad (1.129)$$

For different planets travelling around the same sun of mass M , the ratio k/m is fixed since $k = GMm$, where G is the universal gravitational constant. Hence we see that a^3/T^2 is a constant for a solar system. This is Kepler's third law.

1.8 Dynamics in a Uniformly Rotating Frame

1.8.1 Coordinate Transformation

It is often convenient to work with a coordinate system which is not inertial. This is the case, for example when working on the earth where it is often more convenient to use a frame which is fixed relative to the earth rather than one which is fixed in space. For convenience, let us use two Cartesian coordinate systems, one labelled x_s, y_s, z_s which is fixed in space and the other labelled x_b, y_b, z_b which is fixed with respect to the rotating body (the earth). At $t = 0$, we shall suppose that the two frames are coincident and that the z axis is chosen to be the axis of rotation, so that the angular velocity vector is $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}_s = \omega \hat{\mathbf{z}}_b$. We shall assume that $\boldsymbol{\omega}$ is constant.

The transformation between the unit vectors of the two frames is seen to be

$$\hat{\mathbf{x}}_b = \hat{\mathbf{x}}_s \cos \omega t + \hat{\mathbf{y}}_s \sin \omega t \quad (1.130a)$$

$$\hat{\mathbf{y}}_b = -\hat{\mathbf{x}}_s \sin \omega t + \hat{\mathbf{y}}_s \cos \omega t \quad (1.130b)$$

$$\hat{\mathbf{z}}_b = \hat{\mathbf{z}}_s \quad (1.130c)$$

which may be written in matrix form as

$$\begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \mathbf{A} \quad (1.131)$$

As usual, this defines the transformation between the components of a vector expressed in the two coordinate systems. If

$$\mathbf{G} = \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \mathbf{G}^b = \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \mathbf{G}^s \quad (1.132)$$

then

$$\begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \mathbf{A} \mathbf{G}^b = \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \mathbf{G}^s \quad (1.133)$$

and so $\mathbf{G}^s = \mathbf{A} \mathbf{G}^b$ or $\mathbf{G}^b = \mathbf{A}^{-1} \mathbf{G}^s$.

In order to differentiate vectors expressed in terms of the body frame, we need to know how to differentiate the unit vectors. Of course, the derivatives of the unit vectors fixed in space vanish, and so

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} &= \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \frac{d\mathbf{A}}{dt} = \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \boldsymbol{\omega} \begin{pmatrix} -\sin\omega t & -\cos\omega t & 0 \\ \cos\omega t & -\sin\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \mathbf{A}^{-1} \frac{d\mathbf{A}}{dt} = \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (1.134)$$

From this result, we can find the time-derivative of any vector. For example, for the vector \mathbf{G} above, using the product rule,

$$\begin{aligned} \frac{d\mathbf{G}}{dt} &= \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \frac{d\mathbf{G}^b}{dt} + \frac{d}{dt} \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \mathbf{G}^b \\ &= \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \left\{ \frac{d\mathbf{G}^b}{dt} + \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{G}^b \right\} \\ &= \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \left\{ \frac{d\mathbf{G}^b}{dt} + \boldsymbol{\omega} \times \mathbf{G}^b \right\} \end{aligned} \quad (1.135)$$

where $\boldsymbol{\omega}$ is the column vector $(0, 0, \omega)^t$.

We can also evaluate this derivative in the space frame. This yields

$$\frac{d\mathbf{G}}{dt} = \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \frac{d\mathbf{G}^s}{dt} \quad (1.136)$$

We can equate these two expressions $\dot{\mathbf{G}}$ and use $\begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}}_s & \hat{\mathbf{y}}_s & \hat{\mathbf{z}}_s \end{pmatrix} \mathbf{A}$ to conclude that

$$\mathbf{G}^s = \mathbf{A}\mathbf{G}^b \implies \frac{d\mathbf{G}^s}{dt} = \mathbf{A} \left\{ \frac{d\mathbf{G}^b}{dt} + \boldsymbol{\omega} \times \mathbf{G}^b \right\} \quad (1.137)$$

As a simple application of these results, consider differentiating the position vector \mathbf{r} of the particle. We obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \left\{ \frac{d\mathbf{r}^b}{dt} + \boldsymbol{\omega} \times \mathbf{r}^b \right\} \equiv \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \mathbf{v}^b \quad (1.138)$$

The term in the braces is clearly \mathbf{v}^b , the components of the velocity vector \mathbf{v} in the body coordinate system. A second differentiation gives the acceleration

$$\begin{aligned} \mathbf{a} &= \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \left\{ \frac{d\mathbf{v}^b}{dt} + \boldsymbol{\omega} \times \mathbf{v}^b \right\} \\ &= \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \left\{ \frac{d}{dt} \left[\frac{d\mathbf{r}^b}{dt} + \boldsymbol{\omega} \times \mathbf{r}^b \right] + \boldsymbol{\omega} \times \left[\frac{d\mathbf{r}^b}{dt} + \boldsymbol{\omega} \times \mathbf{r}^b \right] \right\} \\ &= \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \left\{ \ddot{\mathbf{r}}^b + 2(\boldsymbol{\omega} \times \dot{\mathbf{r}}^b) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^b) \right\} \equiv \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \mathbf{a}^b \end{aligned} \quad (1.139)$$

Important Note: The quantities $\dot{\mathbf{r}}^b$ and $\ddot{\mathbf{r}}^b$ are the derivatives of the components of \mathbf{r} in the body frame. They are **not** the components of the derivatives of \mathbf{r} in the body frame. Some books denote $\dot{\mathbf{r}}^b$ by \mathbf{v}_b and $\ddot{\mathbf{r}}^b$ by \mathbf{a}_b but these are different from our \mathbf{v}^b and \mathbf{a}^b above.

1.9 The Law of Motion in a Rotating Frame

Suppose that the particle is acted upon by a force \mathbf{F} . This may be expressed in either the space or body frames and its vector of components in each of these frames may be denoted by \mathbf{F}^s and \mathbf{F}^b respectively.

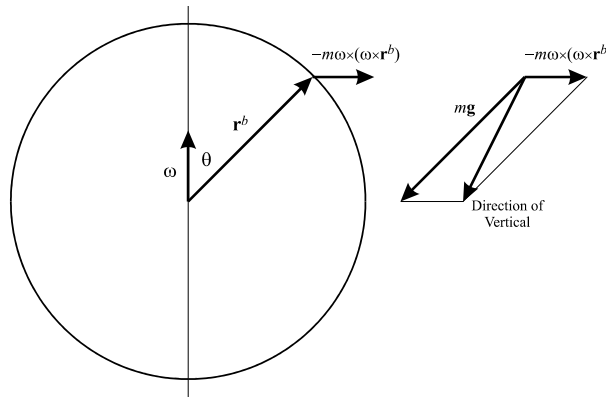


Figure 1.1 Effect of centrifugal force on the direction of the vertical.

Using Newton's second law, we see that

$$\mathbf{F} = (\hat{x}_b \hat{y}_b \hat{z}_b) \mathbf{F}^b = m\mathbf{a} = (\hat{x}_b \hat{y}_b \hat{z}_b) m \left\{ \ddot{\mathbf{r}}^b + 2(\boldsymbol{\omega} \times \dot{\mathbf{r}}^b) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^b) \right\} \quad (1.140)$$

Thus

$$\mathbf{F}^b = m \left\{ \ddot{\mathbf{r}}^b + 2(\boldsymbol{\omega} \times \dot{\mathbf{r}}^b) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^b) \right\} \quad (1.141)$$

or

$$m\ddot{\mathbf{r}}^b = \mathbf{F}^b - 2m(\boldsymbol{\omega} \times \dot{\mathbf{r}}^b) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^b) \quad (1.142)$$

Notice that in the rotating frame, there are two extra terms in Newton's second law. These are due to the fact that we are not working in an inertial frame.

1. $-2m(\boldsymbol{\omega} \times \dot{\mathbf{r}}^b)$ is called the **Coriolis force**. It is zero for a particle which is stationary with respect to the rotating frame ($\dot{\mathbf{r}}^b = 0$). Such a particle is of course moving in the inertial frame, since $\dot{\mathbf{r}} \neq 0$.
2. $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^b)$ is called the **centrifugal force**. It is simply due to the tendency of a particle to remain at rest (or in uniform motion) in an inertial frame as seen from an observer in the rotating frame.

We now consider a series of examples illustrating the use of this equation.

Example: Direction of the vertical

If the earth were not rotating, the direction of the vertical at a point in its surface would be directed towards the centre of the earth. With the rotation, a particle which is stationary on the earth's surface experiences a centrifugal force in addition to the weight force. If θ is the co-latitude of the particle (i.e., θ is the angle measured from the north pole), the centrifugal acceleration is

$$\mathbf{a}_{\text{cent}} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^b)$$

Referring to Figure 1.1, and working in the (x, z) plane for convenience, the vectors $\boldsymbol{\omega}$ and \mathbf{r}^b have the following Cartesian components in the rotating frame

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \quad \text{and} \quad \mathbf{r}^b = \begin{pmatrix} R \sin \theta \\ 0 \\ R \cos \theta \end{pmatrix} \quad (1.143)$$

where R is the radius of the earth. Using the definition of the cross product,

$$-\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} R \sin \theta \\ 0 \\ R \cos \theta \end{pmatrix} \right] = -\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} 0 \\ \omega R \sin \theta \\ 0 \end{pmatrix} = \begin{pmatrix} \omega^2 R \sin \theta \\ 0 \\ 0 \end{pmatrix} \quad (1.144)$$

This force is directed away from the axis of rotation. For the earth, $\omega \approx 7.3 \times 10^{-5} \text{ rad s}^{-1}$ and $R \approx 6.4 \times 10^6 \text{ m}$. Thus $\omega^2 R \approx 0.034 \text{ m s}^{-2}$, which is about 0.3% of the value of gravity. The contribution of the centrifugal force is largest at the equator, but of course, at the equator it serves to reduce the apparent gravity, rather than to change its direction.

Example: Deflection due to Coriolis force

As mentioned above, the Coriolis force acts on a particle which moves relative to the rotating coordinate system. On the earth, this causes a particle such as a bullet fired from a gun to deviate from a straight line in the earth's coordinate system. Consider the coordinate system (x^b, y^b, z^b) on the earth where x^b points eastward, y^b points northwards and z^b points upwards (see Figure 1.2a). We see that ω lies in the (y^b, z^b) plane and has coordinates $(0, \omega \sin \theta, \omega \cos \theta)$. When we draw these axes from the point of view of a person standing on the ground, we see that in the northern hemisphere, the vector ω lies in a vertical plane oriented in the North-South direction and points out of the ground towards the north at an angle of θ from the vertical (see Figure 1.2b).

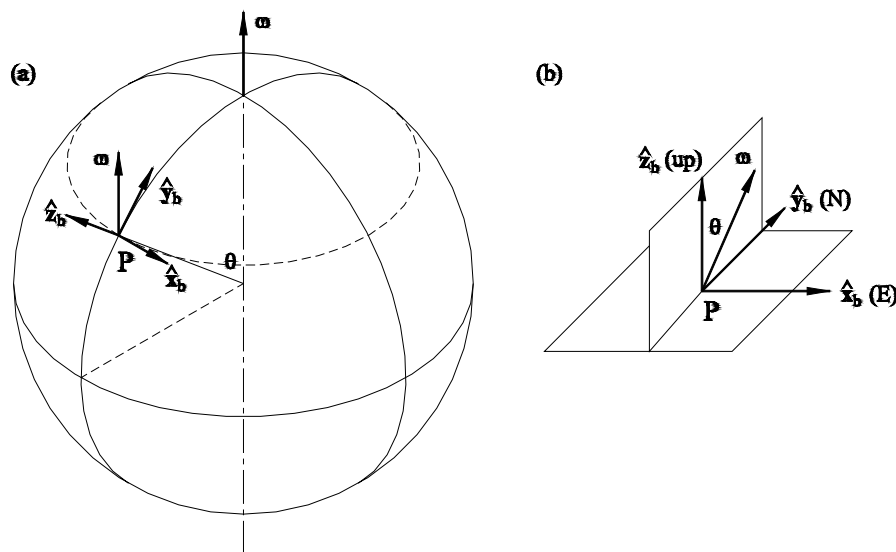


Figure 1.2 Body coordinates on the earth at a point in the northern hemisphere. The angular velocity ω lies in the vertical plane oriented in the North-South direction.

Now suppose that a bullet is being fired to the north at speed u , so that $\dot{\mathbf{r}}^b$ is $(0, u, 0)$. Calculating the Coriolis acceleration $-2\omega \times \dot{\mathbf{r}}^b$ yields $(2u\omega \cos \theta, 0, 0)$. This acceleration is directed towards the east, and so the bullet veers towards the right. On the other hand if the bullet is fired to the south at speed u , so that $\dot{\mathbf{r}}^b$ is $(0, -u, 0)$ and $-2\omega \times \dot{\mathbf{r}}^b = (-2u\omega \cos \theta, 0, 0)$. The Coriolis acceleration is towards the west, and so once again the bullet veers to the right. If fired in other directions, the Coriolis acceleration always causes a deflection to the right, but there can also be a small vertical component as well. In the southern hemisphere on the other hand, the Coriolis force tends to make particles veer to the left in the rotating frame.

It is the Coriolis force which is responsible for deflecting winds from flowing along pressure gradients to flowing (approximately) along the isobars. For a cyclone (low pressure region), where the flow in the absence of the Coriolis force would be inwards, the deflection to the right in the northern hemisphere leads to an anticlockwise circulation, while the deflection to the left in the southern hemisphere leads to a clockwise circulation.

Example: Motion under gravity in a rotating frame

We consider the (earth-based) rotating coordinate system introduced in the previous example where the x axis is directed towards the east, the y axis is towards the north and the z axis is directed upwards. For convenience, we have dropped the subscript b since we do not need to make reference to the space-based coordinates.

The effects of the centrifugal force can be included to a very good approximation by orienting the z axis along the direction of the effective gravity vector, rather than the line through the centre of the earth and by using g_{eff} rather than g in the equations. This allows us to neglect the $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ term in the equation of motion.

Since in these axes $\boldsymbol{\omega} = (0, \omega \sin \theta, \omega \cos \theta)$ where θ is the co-latitude and $\mathbf{g} = (0, 0, -g_{\text{eff}})$ we can write

$$m \frac{d^2 \mathbf{r}}{dt^2} = m \mathbf{g} - 2m \left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} \right) \quad (1.145)$$

in component form as

$$\frac{d^2 x}{dt^2} = -2\omega \left(\frac{dz}{dt} \right) \sin \theta + 2\omega \left(\frac{dy}{dt} \right) \cos \theta \quad (1.146)$$

$$\frac{d^2 y}{dt^2} = -2\omega \left(\frac{dx}{dt} \right) \cos \theta \quad (1.147)$$

$$\frac{d^2 z}{dt^2} = -g_{\text{eff}} + 2\omega \left(\frac{dx}{dt} \right) \sin \theta \quad (1.148)$$

These are the equations governing motion under gravity (e.g., free fall and projectile motion) on the earth.

Let us consider the problem of free fall from the point $(0, 0, h)$ at height h above the earth's surface. Since the particle moves in the z direction, $\dot{z} \neq 0$ and there is a Coriolis force which tends to deflect the particle in the x direction. The velocity in the x direction remains small throughout the fall and so we can neglect \dot{x} . The equations of interest are thus

$$\frac{d^2 x}{dt^2} = -2\omega \left(\frac{dz}{dt} \right) \sin \theta \quad (1.149)$$

$$\frac{d^2 z}{dt^2} = -g_{\text{eff}} \quad (1.150)$$

Integrating the second of these starting from rest at height h yields

$$z = h - \frac{1}{2} g_{\text{eff}} t^2 \quad (1.151)$$

and so

$$\dot{z} = -g_{\text{eff}} t \quad (1.152)$$

Substituting this into the equation of motion for x , we get

$$\frac{d^2 x}{dt^2} = 2\omega g_{\text{eff}} t \sin \theta \quad (1.153)$$

which may be integrated with the initial conditions $x(0) = 0$, $\dot{x}(0) = 0$ to give

$$x(t) = \frac{\omega g_{\text{eff}} t^3}{3} \sin \theta. \quad (1.154)$$

The time it takes for the particle to fall through h is given by setting $z = 0$ and solving for t . The result is

$$t_{\text{fall}} = \sqrt{\frac{2h}{g_{\text{eff}}}} \quad (1.155)$$

The deflection in the x direction at the end of the fall is thus

$$x(t_{\text{fall}}) = \frac{\omega}{3} \sqrt{\frac{8h^3}{g_{\text{eff}}}} \sin \theta. \quad (1.156)$$

For a fall over 100 m at co-latitude 45° (e.g. Paris) on the earth, the deflection is 1.6 cm.

Exercise: If we do not wish to make the approximation that \dot{x} remains small throughout the fall, it is possible to solve the original system of equations using power series or Laplace transforms. Check that

$$x(t) = \left(\frac{1}{3}\omega g_{\text{eff}} t^3 - \frac{1}{15}\omega^3 g_{\text{eff}} t^5 \right) \sin \theta + O(t^7) \quad (1.157)$$

$$y(t) = \left(-\frac{1}{6}\omega^2 g_{\text{eff}} t^4 + \frac{1}{45}\omega^4 g_{\text{eff}} t^6 \right) \cos \theta \sin \theta + O(t^7) \quad (1.158)$$

$$z(t) = h - \frac{1}{2}g_{\text{eff}} t^2 + \left(\frac{1}{6}\omega^2 g_{\text{eff}} t^4 - \frac{1}{45}\omega^4 g_{\text{eff}} t^6 \right) \sin^2 \theta + O(t^7) \quad (1.159)$$

and that the exact solution is

$$x(t) = \frac{g_{\text{eff}}}{4\omega^2} (2\omega t - \sin 2\omega t) \sin \theta \quad (1.160)$$

$$y(t) = \frac{g_{\text{eff}}}{4\omega^2} (1 - \cos 2\omega t - 2\omega^2 t^2) \cos \theta \sin \theta \quad (1.161)$$

$$z(t) = h - \frac{1}{2}g_{\text{eff}} t^2 - \frac{g_{\text{eff}}}{4\omega^2} (1 - \cos 2\omega t - 2\omega^2 t^2) \sin^2 \theta \quad (1.162)$$

Example: The Foucault pendulum

The Foucault pendulum is a long pendulum swinging in the earth's gravitational field. If the oscillation angular frequency is Ω , and the bob swings largely in the xy plane ($z = 0$, since the pendulum is long), the equations of motion in the earth-based coordinate system including the Coriolis acceleration are:

$$\frac{d^2 x}{dt^2} = -\Omega^2 x + 2\omega \left(\frac{dy}{dt} \right) \cos \theta \quad (1.163)$$

$$\frac{d^2 y}{dt^2} = -\Omega^2 y - 2\omega \left(\frac{dx}{dt} \right) \cos \theta \quad (1.164)$$

$$z = 0 \quad (1.165)$$

This system of equations may be solved by introducing the complex variable $u = x + iy$. Taking the first differential equation and adding i times the second, we obtain

$$\frac{d^2}{dt^2} (x + iy) = -\Omega^2 (x + iy) + 2\omega \frac{d}{dt} (y - ix) \cos \theta \quad (1.166)$$

or

$$\frac{d^2 u}{dt^2} = -\Omega^2 u - 2i\omega \frac{du}{dt} \cos \theta \quad (1.167)$$

This is a linear differential equation for u with solutions of the form $\exp(\lambda t)$ where λ is a root of

$$\lambda^2 = -\Omega^2 - 2i\omega \lambda \cos \theta \quad (1.168)$$

The roots are

$$\lambda = -i\omega \cos \theta \pm i\sqrt{\Omega^2 + \omega^2 \cos^2 \theta} \equiv -i\mu \pm i\Omega' \quad (1.169)$$

where $\mu = \omega \cos \theta$ and $\Omega' = \sqrt{\Omega^2 + \omega^2 \cos^2 \theta}$. The general solution for u is

$$u(t) = \exp(-i\mu t) \left(A e^{i\Omega' t} + B e^{-i\Omega' t} \right) \quad (1.170)$$

where the constants A and B depend upon the initial conditions. If we temporarily set the earth's rotation rate to zero $\omega = 0$, we see that $\mu = 0$ and $\Omega' = \Omega$. Then

$$u(t) = x(t) + iy(t) = A e^{i\Omega t} + B e^{-i\Omega t} \quad (1.171)$$

For general A and B , this describes a conical pendulum whose bob swings in an ellipse. If we choose $A = B$ and assume that they are both real, we get

$$u(t) = x(t) + iy(t) = 2A \cos \Omega t \quad (1.172)$$

which corresponds to the bob swinging in the (vertical) xz plane. Including the effects of non-zero ω with these initial conditions, we find

$$u(t) = x(t) + iy(t) = 2A \exp(-i\mu t) \cos \Omega' t \quad (1.173)$$

or

$$x(t) = 2A \cos \Omega' t \cos \mu t \quad (1.174)$$

$$y(t) = -2A \cos \Omega' t \sin \mu t \quad (1.175)$$

The plane of oscillation is now rotating at the angular speed μ which is a slow rate compared to $\Omega' \approx \Omega$. The period of the rotation is $2\pi/\mu = 2\pi/(\omega \cos \theta) = 24 \text{ hours}/\cos \theta$. At the poles, where $\theta = 0$ or 180° , the plane of the pendulum rotates in a day relative to the earth since its plane of oscillation is actually fixed in space. Away from the poles, the plane rotates less than a complete revolution in a day while on the equator there is no rotation at all.