Chapter 2 Systems of Particles and Rigid Bodies

In this chapter, we consider the dynamics of systems of particles. We may choose any portion of the universe and consider that portion to be a "system", leaving the rest of the universe as the "environment" which acts on that system. When considering the forces on the particles making up a system, it is convenient to divide the forces into those due to the environment (the "external" forces) and those which are due to the interactions with other particles in the system (the "internal" forces). When solving problems it is essential to **identify the system** or systems we wish to consider and to **isolate** each system before applying the rules that we shall derive below.

It is also usual in classical mechanics to regard a system as being identified by the particles which make it up, so that particles neither enter nor leave the system. If we wish to solve problems where the object of interest changes in mass (such as a rocket, which ejects matter), the system should be enlarged to include the object of interest together with the mass that it ejects or accretes. An example of such a problem will be considered later.

Conservation laws are of particular importance when studying systems of particles. If there are a large number of particles in the system, it is convenient for us to be able to make general statements about the motion of the system as a whole without having to work through the details of the motion of the individual particles. Perhaps the most important conservation laws are those of linear momentum, angular momentum and of total energy which will be discussed below. In each case, there is a number (or vector) which is a function of the positions and velocities of all the particles of the system which remains fixed throughout the motion of the system.

Let us suppose that our system consists of N particles of masses $m_1, ..., m_N$ located at positions $\mathbf{r}_1(t), ...,$ $\mathbf{r}_N(t)$ at time t relative to some origin O. As for a single particle, we define the velocity $\mathbf{v}_k$, acceleration $\mathbf{a}_k$, momentum $\mathbf{p}_k$, of the k'th particle.The force $\mathbf{F}_k$ on particle k can be divided into two contributions

$$\mathbf{F}_k = \mathbf{F}_k^{(\mathrm{e})} + \mathbf{F}_k^{(\mathrm{i})} \tag{2.1}$$

where $\mathbf{F}_k^{(\mathrm{e})}$ is the external force on particle k due to the environment, and $\mathbf{F}_k^{(\mathrm{i})}$ is the internal force on particle k due to all the **other** particles in the system. We shall further write

$$\mathbf{F}_k^{(\mathrm{i})} = \sum_{j \neq k} \mathbf{F}_{jk} \tag{2.2}$$

where $\mathbf{F}_{jk}$ is the force due to particle j on particle k. For most problems, we shall assume that:

- The forces of interaction satisfy **Newton's third law**, that action and reaction are equal and opposite, i.e.,

  $$\mathbf{F}_{jk} = -\mathbf{F}_{kj} \tag{2.3}$$

  for all distinct indices j and k. This is needed to show that the **linear momentum** of an isolated system of particles is conserved.

- The forces of interaction are **central**, directed along the line joining the two particles, i.e., that

  $$(\mathbf{r}_j - \mathbf{r}_k) \parallel \mathbf{F}_{jk} \text{ or } (\mathbf{r}_j - \mathbf{r}_k) \times \mathbf{F}_{jk} = \mathbf{0} \tag{2.4}$$

  again for all distinct indices j and k. This is needed to show that the **angular momentum** of an isolated system of particles is conserved.

Note that **neither** of these laws hold true in general for systems of moving charges. In Figure 2.1, two positive charges Q_1 and Q_2 are travelling at velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ as shown. The magnetic field due to Q_1 at the position of Q_2 is denoted $\mathbf{B}_1$ and the forces on charge 2 due to charge 1 are denoted by $\mathbf{F}_{12}^{\mathrm{elec}}$ and $\mathbf{F}_{12}^{\mathrm{mag}}$ for the electric and magnetic components respectively. Similarly, the magnetic field due to Q_2 at the position of Q_1 is denoted $\mathbf{B}_2$ and the forces on charge 1 due to charge 2 are denoted by $\mathbf{F}_{21}^{\mathrm{elec}}$ and $\mathbf{F}_{21}^{\mathrm{mag}}$. Although the electric forces $\mathbf{F}_{12}^{\mathrm{elec}}$ and $\mathbf{F}_{21}^{\mathrm{elec}}$ are equal and opposite, and are central forces, the magnetic forces $\mathbf{F}_{12}^{\mathrm{mag}}$ and $\mathbf{F}_{21}^{\mathrm{mag}}$ do not obey these conditions. The centre of mass of the particles accelerates in the

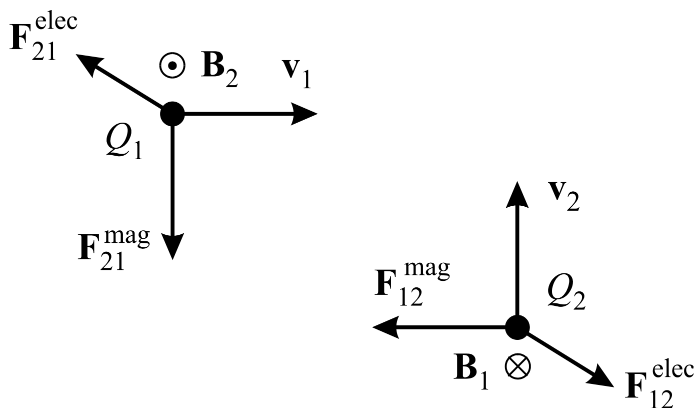

Figure 2.1 Electric and magnetic forces between two moving charges

direction of the total force (which is to the bottom left of the figure) even though there are no external forces on the system. If we consider the particles alone, this means that neither linear nor angular momentum is conserved. In order to recover the conservation laws, we need to include the linear and angular momentum of the electromagnetic field that the charges produce as well. When fields are present, their dynamical properties need to be considered as part of the system.

2.1 Force and Momentum

Newton's second law applies for each particle in the system, and so if $\mathbf{F}_k$ is the **total** force on particle k,

$$\frac{\mathrm{d}\mathbf{p}_k}{\mathrm{d}t} = \mathbf{F}_k \tag{2.5}$$

and we have N vector equations of motion for the system of particles.

Writing $\mathbf{F}_k$ in terms of the external and internal forces,

$$\frac{\mathrm{d}\mathbf{p}_k}{\mathrm{d}t} = \mathbf{F}_k^{(\mathrm{e})} + \sum_{j\neq k} \mathbf{F}_{jk} \tag{2.6}$$

Summing over all the particles, we see that

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \sum_k \frac{\mathrm{d}\mathbf{p}_k}{\mathrm{d}t} = \sum_k \mathbf{F}_k^{(\mathrm{e})} + \sum_k \sum_{j\neq k} \mathbf{F}_{jk} \tag{2.7}$$

By Newton's third law, the internal forces $\mathbf{F}_{jk}$ and $\mathbf{F}_{kj}$ are opposite. This means that the terms in the double sum cancel in pairs, leaving

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \sum_k \mathbf{F}_k^{(\mathrm{e})} \equiv \mathbf{F} \tag{2.8}$$

Thus the rate of change of the **total momentum** depends on the **total external force $\mathbf{F}$** acting on the system. Exactly which particle each force acts upon is unimportant, as are the internal forces.

In particular, if the total external force is zero, the total momentum of the system of particles is conserved, no matter how complicated the internal dynamics of the system may be. The forces of interaction are not restricted to being smooth functions of time. Even if the particles within the system collide with each other and exert impulses on each other, the total momentum is still conserved, provided only that the external force on the system vanishes. The principle of conservation of momentum is thus also useful in problems involving collisions and explosions.

2.1.1 The Centre of Mass

The centre of mass of a system of point particles is defined to be at

$$\mathbf{R} = \frac{\sum_k m_k \mathbf{r}_k}{\sum_k m_k} \tag{2.9}$$

With this definition,

$$M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \left(\sum_k m_k\right)\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \sum_k m_k \frac{\mathrm{d}\mathbf{r}_k}{\mathrm{d}t} = \sum_k \mathbf{p}_k = \mathbf{P} \tag{2.10}$$

where M is the sum of the masses of the particles in the system. This states that the total momentum of the system of particles is equal to the momentum of a point mass of mass M which moves with the centre of mass of the system.

Differentiating this equation and using the previous result,

$$M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \sum_k \mathbf{F}_k^{(\mathrm{e})} \equiv \mathbf{F} \tag{2.11}$$

and so the acceleration of the centre of mass of the system is dependent only upon the total external force acting and the total mass.of the system.

Example: Flexible chain falling onto a table

Consider a flexible chain of length l with linear mass density ρ so that the total mass $M = \rho l$. Initially the chain is held vertically so that the lowest point of the chain is just touching the table. The chain is released and falls onto the table. At time t, suppose that the chain has fallen through a distance s, so that the mass of chain on the table is ρs. Determine the force on the table as the chain falls.

The total force acting on the chain is the sum of $\rho g l$ downwards, due to gravity and R, the reaction force of the table, directed upwards. If we consider the z axis to be directed vertically upwards, the total force is $R - \rho g l$. According to the above, this must be equal to $\rho l \ddot{z}_{\mathrm{cm}}$ where z_{cm} denotes the position of the centre of mass. At time t, the chain consists of a vertical section of mass $\rho(l-s)$ and a section of mass ρs lying on the table. The position of the centre of mass is thus given by

$$z_{\mathrm{cm}} = \frac{\rho(l-s)\times\frac{1}{2}(l-s) + \rho s \times 0}{\rho l} = \frac{(l-s)^2}{2l} \tag{2.12}$$

Since the chain is assumed to be flexible, the top of the chain falls like a free body and so the relationship between s and t is $s = \frac{1}{2}gt^2$. The acceleration of the centre of mass is thus given by

$$\ddot{z}_{\mathrm{cm}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\left(l - \frac{1}{2}gt^2\right)^2}{2l} = \frac{g}{2l}\left(3gt^2 - 2l\right) \tag{2.13}$$

Newton's second law thus reads

$$R - \rho g l = \rho l\left[\frac{g}{2l}\left(3gt^2 - 2l\right)\right] \tag{2.14}$$

from which we see that

$$R = \frac{3}{2}\rho g^2 t^2 = 3\rho s g. \tag{2.15}$$

This is three times the weight of the chain which is on the table at this instant. That this result is reasonable may be seen by considering the rate at which the momentum of the chain is lost as the links hit the table and stop. At time t, the speed of the links of the falling chain which have not yet hit the table is gt, since they fall as free bodies and accelerate uniformly from rest. In a short interval of length $\mathrm{d}t$, the mass hitting the table is thus $\rho(gt)\,\mathrm{d}t$, and the momentum that it carries is $\rho(gt)^2\,\mathrm{d}t$. Since this portion of the chain is brought to rest as it hits the table, this change in momentum is brought about by the impulse due to the reaction of the table. The force due to this effect is thus $\rho(gt)^2$. When this is added to the weight of the chain on the table, namely $\rho s g = \frac{1}{2}\rho(gt)^2$, this gives the total reaction force calculated above.

Example: Motion of a rocket

Consider a rocket which propels itself by ejecting mass in the direction opposite to its motion. If the speed at which the mass is ejected relative to the rocket is w, determine the final velocity v_f, given that it starts from rest, the initial mass of rocket and fuel is M_i and the final mass is M_f.

For this problem, it is important to be clear about what constitutes the system at each stage of the analysis. If we consider the system to be the rocket and all the ejecta, there are no external forces acting on the system and so the centre of mass remains stationary at all times. This is a true but rather unhelpful result. In order to apply the theory, we need a system whose mass is unchanging. Let us consider the interval of time from t to $t+\mathrm{d}t$ and denote the mass of the rocket and the remaining fuel at time t by $M(t)$. The system is chosen to be the rocket together with the material ejected during this interval, so that the mass of the system remains at $M(t)$. At the beginning of the interval, the centre of mass of the system is travelling at velocity $v(t)$. At the end of the interval, the rocket has mass $M(t+\mathrm{d}t)$ and is travelling at velocity $v(t+\mathrm{d}t)$ while the ejecta of mass $M(t)-M(t+\mathrm{d}t)$ is travelling at velocity $v(t)-w$. The momentum of the system at $t+\mathrm{d}t$ is thus

$$M(t+\mathrm{d}t)\,v(t+\mathrm{d}t)+\{M(t)-M(t+\mathrm{d}t)\}\,(v(t)-w)\,. \tag{2.16}$$

Using Taylor's theorem (i.e., $M(t+\mathrm{d}t)=M(t)+\mathrm{d}t\,M'(t)$ etc.), and retaining terms up to first order, the momentum at $t+\mathrm{d}t$ is

$$M(t)\,v(t)+M(t)\,v'(t)\;\mathrm{d}t+M'(t)\,w\mathrm{d}t \tag{2.17}$$

Since no external forces act in this interval, this momentum must be the same as the momentum at time t, which is $M(t)\,v(t)$. Equating these momenta, we see that

$$M(t)\,v'(t)\;\mathrm{d}t+M'(t)\,w\mathrm{d}t=0 \tag{2.18}$$

or

$$v'(t)=-\frac{M'(t)}{M(t)}w. \tag{2.19}$$

This gives the change of velocity in the interval in terms of the amount of matter ejected. During the next interval of time, we consider a slightly different system, as the material ejected during the current interval is no longer considered to be part of the system. The final conditions (i.e., the mass and velocity) taken from the current interval are used as initial conditions for the next, and so equation (2.19) may be regarded as a differential equation for the evolution of M and v for this succession of systems. Integrating the equation from the initial to the final time yields

$$v_f-0=-w\int_{t_i}^{t_f}\frac{M'(t)}{M(t)}\mathrm{d}t=-w\left[\log M(t)\right]_{t_i}^{t_f}=w\log\frac{M_i}{M_f}. \tag{2.20}$$

It should be carefully noted how we used the conservation of momentum only when we were considering a system of constant mass. During the process of relabelling the system to discard the ejecta that was ejected during the previous time interval, the momentum of the new system is different from the momentum of the old. The results of the calculation from the previous interval only serve to specify the initial conditions for the new system.

2.2 Total Angular Momentum

Given an origin O, the total angular momentum $\mathbf{L}$ of a system of particles is the sum of all the individual angular momenta. Its rate of change is given by

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \sum_{k=1}^{N}\frac{\mathrm{d}\mathbf{L}_k}{\mathrm{d}t}=\sum_{k=1}^{N}\mathbf{r}_k\times\mathbf{F}_k=\sum_{k=1}^{N}\mathbf{r}_k\times\left(\mathbf{F}_k^{(\mathrm{e})}+\sum_{j\neq k}\mathbf{F}_{jk}\right)\\ &= \sum_{k=1}^{N}\mathbf{r}_k\times\mathbf{F}_k^{(\mathrm{e})}+\sum_{k=1}^{N}\sum_{j\neq k}\mathbf{r}_k\times\mathbf{F}_{jk}\end{aligned} \tag{2.21}$$

The first summation is $\mathbf{N}_{\rm tot}^{(e)}$, the total torque acting on the system about O due to the external forces. The second summation can be shown to vanish for central forces satisfying Newton's third law since

$$\begin{aligned}\sum_{k=1}^{N}\sum_{j\neq k}\mathbf{r}_k\times\mathbf{F}_{jk} &= \frac{1}{2}\sum_{k=1}^{N}\sum_{j\neq k}\left(\mathbf{r}_k\times\mathbf{F}_{jk}+\mathbf{r}_j\times\mathbf{F}_{kj}\right)\\ &= \frac{1}{2}\sum_{k=1}^{N}\sum_{j\neq k}\left(\mathbf{r}_k\times\mathbf{F}_{jk}-\mathbf{r}_j\times\mathbf{F}_{jk}\right)\text{ by Newton's third law}\\ &= \frac{1}{2}\sum_{k=1}^{N}\sum_{j\neq k}\left(\mathbf{r}_k-\mathbf{r}_j\right)\times\mathbf{F}_{jk}=0\text{ since the forces are central}\end{aligned}\tag{2.22}$$

Thus,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}=\sum_{k=1}^{N}\mathbf{r}_k\times\mathbf{F}_k^{(e)}=\mathbf{N}_{\rm tot}^{(e)}\tag{2.23}$$

If the total external torque vanishes, the total angular momentum is conserved.

2.3 Decomposition into Translational and Rotational Motions

We shall next show that the motion of a system of particles can profitably be thought of in terms of the motion **of** the centre of mass together with motion **about** the centre of mass. The former may be regarded as the translational motion while the latter is the rotational motion.

In order to carry out the decomposition, we consider the positions of the particles relative to the centre of mass $\mathbf{R}(t)$, to this end, we define

$$\mathbf{r}_k'(t)=\mathbf{r}_k(t)-\mathbf{R}(t)\tag{2.24}$$

Similarly, we define the velocities relative to the centre of mass

$$\dot{\mathbf{r}}_k'(t)=\dot{\mathbf{r}}_k(t)-\dot{\mathbf{R}}(t)\text{ or }\mathbf{v}_k'(t)=\mathbf{v}_k(t)-\mathbf{V}(t).\tag{2.25}$$

The total angular momentum may be written as

$$\begin{aligned}\mathbf{L} &= \sum_{k=1}^{N}m_k\mathbf{r}_k\times\mathbf{v}_k=\sum_{k=1}^{N}m_k\left(\mathbf{R}+\mathbf{r}_k'\right)\times\left(\mathbf{V}+\mathbf{v}_k'\right)\\ &= \sum_{k=1}^{N}m_k\left(\mathbf{R}\times\mathbf{V}\right)+\left(\mathbf{R}\times\sum_{k=1}^{N}m_k\mathbf{v}_k'\right)+\left(\sum_{k=1}^{N}m_k\mathbf{r}_k'\times\mathbf{V}\right)+\sum_{k=1}^{N}m_k\left(\mathbf{r}_k'\times\mathbf{v}_k'\right)\end{aligned}\tag{2.26}$$

By the definition of the centre of mass, $\sum_{k=1}^{N}m_k\mathbf{r}_k'$ vanishes identically at all times, and so $\sum_{k=1}^{N}m_k\mathbf{v}_k'=0$ as well. Thus

$$\mathbf{L}=M\left(\mathbf{R}\times\mathbf{V}\right)+\sum_{k=1}^{N}m_k\left(\mathbf{r}_k'\times\mathbf{v}_k'\right)\tag{2.27}$$

The first term is the angular momentum about O of a particle of mass M moving with the centre of mass. The second term is the angular momentum of the system about the centre of mass. This can be a convenient way of computing $\mathbf{L}$.

Calculating the time-derivative of $\mathbf{L}$,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= M\left(\dot{\mathbf{R}}\times\mathbf{V}+\mathbf{R}\times\dot{\mathbf{V}}\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{k=1}^{N}m_k\left(\mathbf{r}_k'\times\mathbf{v}_k'\right)\right\}\\ &= \mathbf{R}\times\mathbf{F}_{\rm tot}^{(e)}+\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{k=1}^{N}m_k\left(\mathbf{r}_k'\times\mathbf{v}_k'\right)\right\}\end{aligned}\tag{2.28}$$

Combining this with (2.23), we obtain

$$\sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^{(\mathrm{e})} = \mathbf{R} \times \mathbf{F}_{\mathrm{tot}}^{(\mathrm{e})} + \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \sum_{k=1}^{N} m_k \left(\mathbf{r}'_k \times \mathbf{v}'_k \right) \right\} \tag{2.29}$$

which may be rearranged to give

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \sum_{k=1}^{N} m_k \left(\mathbf{r}'_k \times \mathbf{v}'_k \right) \right\} = \sum_{k=1}^{N} \left(\mathbf{r}_k - \mathbf{R} \right) \times \mathbf{F}_k^{(\mathrm{e})} = \sum_{k=1}^{N} \mathbf{r}'_k \times \mathbf{F}_k^{(\mathrm{e})} \tag{2.30}$$

Thus, the rate of change of angular momentum about the centre of mass is given by the sum of the torques due to the external forces also calculated about the centre of mass.

The motion of the system has thus been decomposed into:

- The motion **of** the centre of mass, satisfying

$$M\ddot{\mathbf{R}} = M\dot{\mathbf{V}} = \mathbf{F}_{\mathrm{tot}}^{(\mathrm{e})} \tag{2.31}$$

- The motion **about** the centre of mass, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \sum_{k=1}^{N} m_k \left(\mathbf{r}'_k \times \mathbf{v}'_k \right) \right\} = \sum_{k=1}^{N} \mathbf{r}'_k \times \mathbf{F}_k^{(\mathrm{e})} \tag{2.32}$$

This is known as the **principle of moments.** Note that when we compute the torques and angular momentum about the **centre of mass**, the relationship (2.32) holds even though the centre of mass of the body is accelerating. We cannot do this in general about points other than the centre of mass.

2.4 Kinetic Energy

The total kinetic energy of a system of particles is given by

$$\begin{aligned} T &= \sum_{k=1}^{N} \frac{1}{2} m_k v_k^2 = \sum_{k=1}^{N} \frac{1}{2} m_k \left(\mathbf{v}_k \cdot \mathbf{v}_k \right) = \sum_{k=1}^{N} \frac{1}{2} m_k \left(\mathbf{v}'_k + \mathbf{V} \right) \cdot \left(\mathbf{v}'_k + \mathbf{V} \right) \\ &= \frac{1}{2} \left(\sum_{k=1}^{N} m_k \right) V^2 + \left(\sum_{k=1}^{N} m_k \mathbf{v}'_k \right) \cdot \mathbf{V} + \frac{1}{2} \sum_{k=1}^{N} m_k \left(v'_k \right)^2 \end{aligned} \tag{2.33}$$

By the definition of the centre of mass, $\sum_{k=1}^{N} m_k \mathbf{r}'_k$ vanishes identically at all times, and so $\sum_{k=1}^{N} m_k \mathbf{v}'_k = 0$ as well. Thus

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{k=1}^{N} m_k \left(v'_k \right)^2 \tag{2.34}$$

which means that the total kinetic energy can also be divided into the kinetic energy of the motion of the centre of mass (translational kinetic energy) and of the motion about the centre of mass (rotational kinetic energy).

2.5 The Work-Energy Theorem

The work-energy theorem is found simply by adding the contributions from each particle. If we bring the system from configuration i to configuration f, the change in total kinetic energy is

$$\begin{aligned} T[f]-T[i] &= \sum_{k=1}^{N} T_k[f]-T_k[i]=\sum_{k=1}^{N}\int_i^f \mathbf{F}_k\cdot \mathrm{d}\mathbf{r}_k \\ &= \int_i^f \sum_{k=1}^{N}\left(\mathbf{F}_k^{(\mathrm{e})}+\sum_{j\neq k}\mathbf{F}_{jk}\right)\cdot \mathrm{d}\mathbf{r}_k \end{aligned} \tag{2.35}$$

This expression cannot be simplified in general, since both internal and external forces contribute to the work done. However, if the external forces and forces of interaction are **conservative** so that they are derivable from a potential energy function, a simplification is possible.

In order to avoid obscuring the issues by the notation, we shall consider a specific example of a conservative system. Suppose that we have three particles of masses m_1, m_2, m_3 in a vertical plane at coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). The particles are connected by three springs with spring constants k_a, k_b and k_c. Spring a connects masses 2 and 3, b connects 1 and 3 while c connects 1 and 2. The potential energy of the system is the sum of the gravitational and the elastic potential energies:

$$V(x_1,y_1,x_2,y_2,x_3,y_3)=m_1gy_1+m_2gy_2+m_3gy_3+\frac{1}{2}k_ae_a^2+\frac{1}{2}k_be_b^2+\frac{1}{2}k_ce_c^2 \tag{2.36}$$

where the elastic potential energies have been written in terms of the extensions of the springs, so that for example,

$$e_a=\sqrt{(x_2-x_3)^2+(y_2-y_3)^2}-l_a, \tag{2.37}$$

l_a being the natural length of spring a.

Let us now consider the negative gradient of V with respect to the three position vectors $\mathbf{r}_1=(x_1,y_1)$, $\mathbf{r}_2=(x_2,y_2)$ and $\mathbf{r}_3=(x_3,y_3)$. We see that

$$-\frac{\partial V}{\partial x_1}=-k_ae_a\frac{\partial e_a}{\partial x_1}-k_be_b\frac{\partial e_b}{\partial x_1}-k_ce_c\frac{\partial e_c}{\partial x_1} \tag{2.38}$$

Carrying out the differentiations,

$$-k_be_b\frac{\partial e_b}{\partial x_1} = -k_be_b\frac{(x_1-x_3)}{\sqrt{(x_1-x_3)^2+(y_1-y_3)^2}} \tag{2.39}$$

$$-k_ce_c\frac{\partial e_c}{\partial x_1} = -k_ce_c\frac{(x_1-x_2)}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}} \tag{2.40}$$

These are precisely the x components of the forces which the springs b and c connected to m_1 exert upon the mass. Similarly,

$$-\frac{\partial V}{\partial y_1}=-m_1g-k_ae_a\frac{\partial e_a}{\partial y_1}-k_be_b\frac{\partial e_b}{\partial y_1}-k_ce_c\frac{\partial e_c}{\partial y_1} \tag{2.41}$$

where

$$-k_be_b\frac{\partial e_b}{\partial y_1} = -k_be_b\frac{(y_1-y_3)}{\sqrt{(x_1-x_3)^2+(y_1-y_3)^2}} \tag{2.42}$$

$$-k_ce_c\frac{\partial e_c}{\partial y_1} = -k_ce_c\frac{(y_1-y_2)}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}} \tag{2.43}$$

which are the y components of the forces which the springs b and c connected to m_1 exert upon the mass. The negative gradient with respect to the coordinates of the first mass is thus

$$\begin{aligned} -\boldsymbol{\nabla}_{\mathbf{r}_1} V &= -\frac{\partial V}{\partial x_1}\hat{\mathbf{x}} - \frac{\partial V}{\partial y_1}\hat{\mathbf{y}} = -m_1 g\hat{\mathbf{y}} - k_c e_c (\widehat{\mathbf{r}_1 - \mathbf{r}_2}) - k_b e_b (\widehat{\mathbf{r}_1 - \mathbf{r}_3}) \\ &= \mathbf{F}_1^{(\mathrm{e})} + \mathbf{F}_{21} + \mathbf{F}_{31} \end{aligned} \tag{2.44}$$

where $\mathbf{F}_1^{(\mathrm{e})}$ is the external force of gravity and $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$ are the internal forces due to particles 2 and 3 respectively. We see that the potential energy function has the property that its negative gradient with respect to each of the particle coordinates gives the **total force** (external and internal) acting on that particle, i.e.,

$$\mathbf{F}_k = -\boldsymbol{\nabla}_{\mathbf{r}_k} V \tag{2.45}$$

The work-energy theorem for a **conservative** system of particles is thus

$$T[f] - T[i] = -\sum_{k=1}^{N} \int_i^f \boldsymbol{\nabla}_{\mathbf{r}_k} V \cdot \mathrm{d}\mathbf{r}_k = V[i] - V[f] \tag{2.46}$$

This may be rearranged to yield the principle of conservation of the total energy

$$T[f] + V[f] = T[i] + V[i] \tag{2.47}$$

In general, the total potential energy $V(\mathbf{r}_1, ..., \mathbf{r}_N)$ contains contributions from the external forces and from the internal forces. If we restrict ourselves to pairwise interactions, we have

$$V(\mathbf{r}_1, ..., \mathbf{r}_N) = \sum_{k=1}^{N} V_k^{(\mathrm{e})}(\mathbf{r}_k) + \sum_{k=1}^{N} \sum_{j<k} V_{jk}(\mathbf{r}_j, \mathbf{r}_k) \tag{2.48}$$

where the second sum is over the **distinct** pairs of particle indices j and k. If the force of interaction is central and obeys Newton's third law,

$$V_{jk}(\mathbf{r}_j, \mathbf{r}_k) = V_{jk}(\|\mathbf{r}_j - \mathbf{r}_k\|). \tag{2.49}$$

The second sum in the potential energy is sometimes taken over all $j \neq k$, rather than $j < k$. This overcounts each interaction term twice and so we have to write

$$V(\mathbf{r}_1, ..., \mathbf{r}_N) = \sum_{k=1}^{N} V_k^{(\mathrm{e})}(\mathbf{r}_k) + \frac{1}{2}\sum_{k=1}^{N} \sum_{j\neq k} V_{jk}(\mathbf{r}_j, \mathbf{r}_k) \tag{2.50}$$

where, of course, $V_{jk} = V_{kj}$.

2.6 Kinematics of Rigid Body Motion

Rigid bodies are systems of particles which are held together by forces which keep the relative distances between pairs of particles fixed. It may be convenient initially to think in terms of a few point masses connected by light inextensible rods, but our considerations will also apply to rigid bodies consisting of a continuous distribution of matter. In previous work, we have considered rotation about a fixed axis, which leads to considerable simplifications, but we would now like to work towards considering more general rotations in which only a **single point** rather than an axis is fixed. More generally, a rigid body can translate as well as rotate, but as we have seen, it is possible to separate out the translational motion of the centre of mass from the motion of the body about the centre of mass so that there is in fact no loss of generality in considering only rotations about a fixed point. This fixed point will be chosen as the origin of our coordinate system. It is usually (but not always) chosen to be at the centre of mass of the body.

The assumption of rigidity leads to a considerable reduction in the number of degrees of freedom in the problem. Normally for a set of N particles, there would be $3N$ degrees of freedom, and fixing the position of the centre of mass reduces this to $3N-3$. For a rigid body, the further constraint that all interparticle distances remain unchanged during the motion reduces the number of degrees of freedom to three. These numbers are all that are needed to specify the orientation of the body about the point O. Once these numbers are known, the locations of all N particles may be written down. Before considering the general case, let us review the problem of rotation about a **fixed** axis for which there is only a single degree of freedom, namely the angle of rotation

2.6.1 Angular Velocity and Angular Momentum

Let us use the notation $\mathbf{r}_k(t)$ for the position of the k'th mass of mass m_k at time t. If the angular velocity of the body is $\boldsymbol{\omega}$, each particle moves in a circle about the fixed axis which is along the direction of $\boldsymbol{\omega}$. The velocity of the k'th particle is thus

$$\mathbf{v}_k = \boldsymbol{\omega} \times \mathbf{r}_k \tag{2.51}$$

where the cross product ensures that the speed depends on the perpendicular distance from the point $\mathbf{r}_k$ to the axis of rotation. The total angular momentum is then given by

$$\mathbf{L} = \sum_{k=1}^{N} m_k (\mathbf{r}_k \times \mathbf{v}_k) = \sum_{k=1}^{N} m_k \mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k) \tag{2.52}$$

We now wish to consider the relationship between the total angular momentum $\mathbf{L}$ and the angular velocity $\boldsymbol{\omega}$. For translational motion, the momentum is simply the **mass** multiplied by the velocity. For rotational motion, it is clear that the relationship (2.52) is more complicated. If we use the Cartesian components of the vectors (which we shall denote by the superscript c),

$$\begin{aligned}
\mathbf{L}^{\mathrm{c}} &= \sum_k m_k \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} \times \left[\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} \right] \\
&= \sum_k m_k \begin{pmatrix} \left(y_k^2+z_k^2\right)\omega_x - x_k y_k \omega_y - x_k z_k \omega_z \\ -y_k x_k \omega_x + \left(z_k^2+x_k^2\right)\omega_y - y_k z_k \omega_z \\ -z_k x_k \omega_x - z_k y_k \omega_y + \left(x_k^2+y_k^2\right)\omega_z \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \sum_k m_k \left(y_k^2+z_k^2\right) & -\sum_k m_k x_k y_k & -\sum_k m_k x_k z_k \\ -\sum_k m_k y_k x_k & \sum_k m_k \left(z_k^2+x_k^2\right) & -\sum_k m_k y_k z_k \\ -\sum_k m_k z_k x_k & -\sum_k m_k z_k y_k & \sum_k m_k \left(x_k^2+y_k^2\right) \end{pmatrix}}_{\mathbf{I}^{\mathrm{c}}} \underbrace{\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}_{\boldsymbol{\omega}^{\mathrm{c}}}
\end{aligned} \tag{2.53}$$

From this expression we see that the angular momentum is **linearly** related to the angular velocity, but instead of the relationship being a simple scalar factor, there is a symmetric matrix $\mathbf{I}^{\mathrm{c}}$ which the Cartesian representation of an object called the "inertia tensor" which relates the two vectors., i.e.,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \tag{2.54}$$

where the Cartesian representation is

$$\mathbf{I}^{\mathrm{c}} = \sum_k m_k \begin{pmatrix} y_k^2+z_k^2 & -x_k y_k & -x_k z_k \\ -y_k x_k & z_k^2+x_k^2 & -y_k z_k \\ -z_k x_k & -z_k y_k & x_k^2+y_k^2 \end{pmatrix} \tag{2.55}$$

Notice that we can also write this as

$$\mathbf{I}^{\mathrm{c}} = \sum_k m_k \left\{ \begin{pmatrix} r_k^2 & 0 & 0 \\ 0 & r_k^2 & 0 \\ 0 & 0 & r_k^2 \end{pmatrix} - \begin{pmatrix} x_k^2 & x_k y_k & x_k z_k \\ y_k x_k & y_k^2 & y_k z_k \\ z_k x_k & z_k y_k & z_k^2 \end{pmatrix} \right\} \tag{2.56}$$

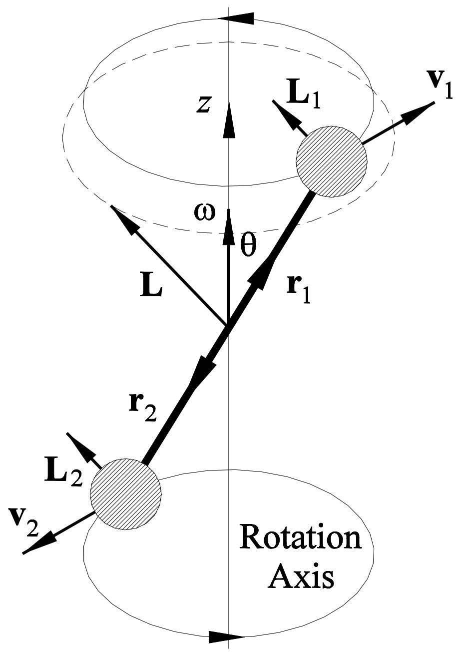

Figure 2.2 Angular velocity $\boldsymbol{\omega}$ and angular momentum $\mathbf{L}$ of a rotating dumbbell. Dashed line shows path of endpoint of $\mathbf{L}$ vector as the system rotates.

where $r_k^2 = x_k^2 + y_k^2 + z_k^2 = \mathbf{r}_k^{\mathrm{t}}\mathbf{r}_k$. The second matrix is just the product $\mathbf{r}_k\mathbf{r}_k^{\mathrm{t}}$ where we regard $\mathbf{r}_k$ as a column vector and its transpose as a row vector. Hence,

$$\mathbf{I} = \sum_k m_k \left(\left(\mathbf{r}_k^{\mathrm{t}}\mathbf{r}_k \right) \mathbf{1} - \mathbf{r}_k\mathbf{r}_k^{\mathrm{t}} \right) \tag{2.57}$$

where $\mathbf{1}$ denotes the 3×3 identity matrix. This alternative form may also be derived by expanding the vector triple product in (2.52):

$$\mathbf{L} = \sum_k m_k \mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k) = \sum_k m_k \left\{ \boldsymbol{\omega} \left(\mathbf{r}_k \cdot \mathbf{r}_k \right) - \mathbf{r}_k \left(\mathbf{r}_k \cdot \boldsymbol{\omega} \right) \right\}$$

which is equivalent to (2.57). Since $\mathbf{I}$ is not a scalar, the directions of $\mathbf{L}$ and $\boldsymbol{\omega}$ will be different in general. This contrasts to the situation for translational motion where $\mathbf{p} = M\mathbf{v}$ for which $\mathbf{p}$ and $\mathbf{v}$ are always parallel.

Example: Consider the angular momentum of the dumbbell which is constrained to rotate at constant angular velocity ω about an axis through its centre of mass as shown in Figure 2.2. The separation of the masses M in the dumbbell is s and the angle between the dumbbell axis and the rotation axis is θ. Calculate the torque required to keep the dumbbell carrying out this motion.

Let the rotation axis be along the z axis. At time t, the positions of the masses in Cartesian coordinates are

$$\mathbf{r}_1(t) = \frac{s}{2} \begin{pmatrix} \sin\theta\cos\omega t \\ \sin\theta\sin\omega t \\ \cos\theta \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2(t) = \frac{s}{2} \begin{pmatrix} -\sin\theta\cos\omega t \\ -\sin\theta\sin\omega t \\ -\cos\theta \end{pmatrix} \tag{2.58}$$

The momenta of the particles are

$$\mathbf{p}_1(t) = \frac{Ms}{2} \begin{pmatrix} -\omega\sin\theta\sin\omega t \\ \omega\sin\theta\cos\omega t \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{p}_2(t) = \frac{Ms}{2} \begin{pmatrix} \omega\sin\theta\sin\omega t \\ -\omega\sin\theta\cos\omega t \\ 0 \end{pmatrix} \tag{2.59}$$

and the angular momenta are

$$\mathbf{L}_1(t) = \mathbf{L}_2(t) = \frac{Ms^2\omega\sin\theta}{4} \begin{pmatrix} -\cos\theta\cos\omega t \\ -\cos\theta\sin\omega t \\ \sin\theta \end{pmatrix} \tag{2.60}$$

and so the total angular momentum is

$$\mathbf{L}(t) = \frac{Ms^2\omega\sin\theta}{2}\begin{pmatrix} -\cos\theta\cos\omega t \\ -\cos\theta\sin\omega t \\ \sin\theta \end{pmatrix} \tag{2.61}$$

Although the angular velocity is a constant and is parallel to the z axis at all times, the angular momentum is varying and precesses about the rotation axis. The varying angular momentum means that a torque is required for this motion. Its value is:

$$\mathbf{N}(t) = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{Ms^2\omega^2\sin 2\theta}{4}\begin{pmatrix} \sin\omega t \\ -\cos\omega t \\ 0 \end{pmatrix} \tag{2.62}$$

This torque is supplied by the bearings holding the shaft defining the rotation axis. Notice that this torque vanishes if $\theta = 0$ or if $\theta = 90°$. As we shall see later, at these angles, the rotation axis coincides with a **principal axis** of the system. When uniform rotation takes place along a principal axis, the angular momentum is constant, no torques are required and there is no tendency for the shaft to wobble. When a wheel for a car is aligned, extra masses are added near the rim of the wheel so that a principal axis is made to coincide with the rotation axis.

It is also possible to carry out the above calculation using the inertia tensor. Using the definition (2.57) and the explicit formulae for the positions of the masses, we find

$$\mathbf{I}^{\mathrm{c}}(t) = \frac{1}{2}Ms^2\begin{pmatrix} 1-\sin^2\theta\cos^2\omega t & -\sin^2\theta\cos\omega t\sin\omega t & -\sin\theta\cos\omega t\cos\theta \\ -\sin^2\theta\cos\omega t\sin\omega t & 1-\sin^2\theta\sin^2\omega t & -\sin\theta\sin\omega t\cos\theta \\ -\sin\theta\cos\omega t\cos\theta & -\sin\theta\sin\omega t\cos\theta & 1-\cos^2\theta \end{pmatrix}. \tag{2.63}$$

This is **time-dependent** since it rotates along with the body. We can apply $\mathbf{I}$ to the angular velocity to find the angular momentum:

$$\mathbf{L}^{\mathrm{c}} = \mathbf{I}^{\mathrm{c}}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \frac{1}{2}Ms^2\begin{pmatrix} -(\cos\theta)\,\omega\sin\theta\cos\omega t \\ -(\cos\theta)\,\omega\sin\theta\sin\omega t \\ \omega\sin^2\theta \end{pmatrix} \tag{2.64}$$

which coincides with the previous result.

2.6.2 Inertia Tensor for a Continuous Distribution of Mass

From the definition of the inertia tensor (2.55), we see that for a system of particles, it is necessary to compute the six quantities

$$\sum_i m_i x_i^2,\quad \sum_i m_i y_i^2,\quad \sum_i m_i z_i^2,\quad \sum_i m_i x_i y_i,\quad \sum_i m_i x_i z_i \quad\text{and}\quad \sum_i m_i y_i z_i \tag{2.65}$$

For a body with a distribution of density $\rho(x, y, z)$, these sums turn into the volume integrals

$$\int_V x^2\rho\,\mathrm{d}V,\quad \int_V y^2\rho\,\mathrm{d}V,\quad \int_V z^2\rho\,\mathrm{d}V,\quad \int_V xy\rho\,\mathrm{d}V,\quad \int_V xz\rho\,\mathrm{d}V \quad\text{and}\quad \int_V yz\rho\,\mathrm{d}V \tag{2.66}$$

where $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and the integral is over the volume occupied by the body. The coordinate axes may be chosen arbitrarily so long that the origin is at the centre of rotation, but it is most convenient to choose them to be fixed in the body. Once we have the components of the inertia tensor $\mathbf{I}$ in this coordinate system, we shall be able to transform it into any other coordinate system about the same centre, such as to the coordinate system which is fixed in space.

Example: Inertia tensor for a uniform sphere about its centre

Suppose that the sphere is of radius R and that the density of the material is ρ. It is convenient to use spherical polar coordinates for which $\mathrm{d}V = r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$. Since $z = r\cos\theta$, we find that

$$\begin{aligned}\int z^2 \rho \,\mathrm{d}V &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho \left(r^2\cos^2\theta\right) r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \rho \left(\int_0^{2\pi} \mathrm{d}\phi\right)\left(\int_0^{\pi} \left(\cos^2\theta\right)(\sin\theta)\,\mathrm{d}\theta\right)\left(\int_0^{R} r^4\,\mathrm{d}r\right) \\ &= \frac{4}{15}\rho\pi R^5 = \frac{1}{5} m R^2 \end{aligned} \tag{2.67}$$

where $m = \frac{4}{3}\pi R^3 \rho$ is the mass of the sphere. By symmetry (check these by doing the integrals),

$$\int x^2 \rho \,\mathrm{d}V = \int y^2 \rho \,\mathrm{d}V = \frac{1}{5} m R^2 \tag{2.68}$$

The other three integrals involving xy, xz and yz vanish by symmetry. The inertia tensor is given by the continuous analogue of (2.55),

$$\mathbf{I}^{\mathrm{c}} = \begin{pmatrix} \int \left(y^2+z^2\right)\rho\,\mathrm{d}V & -\int xy\rho\,\mathrm{d}V & -\int xz\rho\,\mathrm{d}V \\ -\int yx\rho\,\mathrm{d}V & \int \left(z^2+x^2\right)\rho\,\mathrm{d}V & -\int yz\rho\,\mathrm{d}V \\ -\int zx\rho\,\mathrm{d}V & -\int zy\rho\,\mathrm{d}V & \int \left(x^2+y^2\right)\rho\,\mathrm{d}V \end{pmatrix} = \frac{2}{5} m R^2 \mathbf{1} \tag{2.69}$$

where, as expected, the inertia tensor is a multiple of the identity for a uniform sphere.

Example: Inertia tensor for a uniform rectangular block $|x| \le \frac{1}{2}a$, $|y| \le \frac{1}{2}b$, $|z| \le \frac{1}{2}c$ about the origin.

Let the density of the block be ρ so that its mass is $m = \rho abc$. We find that

$$\int x^2 \rho\,\mathrm{d}V = \rho\left(\int_{-a/2}^{a/2} x^2 \mathrm{d}x\right)\left(\int_{-b/2}^{b/2} \mathrm{d}y\right)\left(\int_{-c/2}^{c/2} \mathrm{d}z\right) = \frac{1}{12}\rho a^3 bc = \frac{ma^2}{12} \tag{2.70}$$

and similarly

$$\int y^2 \rho\,\mathrm{d}V = \frac{mb^2}{12}, \quad \int z^2 \rho\,\mathrm{d}V = \frac{mc^2}{12} \tag{2.71}$$

The other integrals are all zero. The inertia tensor is thus diagonal with

$$\mathbf{I}^{\mathrm{c}} = \begin{pmatrix} \frac{m}{12}\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{m}{12}\left(c^2+a^2\right) & 0 \\ 0 & 0 & \frac{m}{12}\left(a^2+b^2\right) \end{pmatrix}. \tag{2.72}$$

2.6.3 Transformation of the Inertia Tensor

In the Cartesian coordinate system, we have the relation

$$\mathbf{L}^{\mathrm{c}} = \mathbf{I}^{\mathrm{c}} \boldsymbol{\omega}^{\mathrm{c}} \tag{2.73}$$

where the relationships between the components and the physical vector are given, as usual by

$$\mathbf{L} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \mathbf{L}^{\mathrm{c}} \text{ and } \boldsymbol{\omega} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \boldsymbol{\omega}^{\mathrm{c}} \tag{2.74}$$

If we change to a new coordinate system with basis vectors $\begin{pmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} & \hat{\mathbf{w}} \end{pmatrix}$ which are related to the old by the transformation matrix $\mathbf{T}$, i.e.,

$$\begin{pmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} & \hat{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \mathbf{T}, \tag{2.75}$$

and if we denote the components with respect to the new coordinate system by the superscript n, then as discussed in the previous chapter, the transformation for the components is given by

$$\mathbf{L}^{\mathrm{n}} = \mathbf{T}^{-1}\mathbf{L}^{\mathrm{c}} \text{ and } \boldsymbol{\omega}^{\mathrm{n}} = \mathbf{T}^{-1}\boldsymbol{\omega}^{\mathrm{c}}. \tag{2.76}$$

Thus we can write

$$\mathbf{L}^{\mathrm{n}} = \mathbf{T}^{-1}\mathbf{L}^{\mathrm{c}} = \mathbf{T}^{-1}\mathbf{I}^{\mathrm{c}}\boldsymbol{\omega}^{\mathrm{c}} = \underbrace{\mathbf{T}^{-1}\mathbf{I}^{\mathrm{c}}\mathbf{T}}_{\mathbf{I}^{\mathrm{n}}}\boldsymbol{\omega}^{\mathrm{n}} \tag{2.77}$$

From which we see that under this coordinate transformation, we have to transform the components of the tensor according to the rule

$$\mathbf{I}^{\mathrm{n}} = \mathbf{T}^{-1}\mathbf{I}^{\mathrm{c}}\mathbf{T} \tag{2.78}$$

in order for the tensor to act as a linear mapping between vectors. If the bases are in fact orthonormal, $\mathbf{T}$ is an orthogonal matrix and $\mathbf{T}^{-1} = \mathbf{T}^{\mathrm{t}}$ and we can write $\mathbf{I}^{\mathrm{n}} = \mathbf{T}^{\mathrm{t}}\mathbf{I}^{\mathrm{c}}\mathbf{T}$. It should be apparent that the above argument applies to any tensor.

2.6.4 Particle Velocity and Angular Velocity for a Rigid System

We have seen from (2.51) that for a rigid body, all the particle velocities are determined once $\boldsymbol{\omega}$ is given

$$\mathbf{v}_k(t) = \boldsymbol{\omega}(t) \times \mathbf{r}_k(t) \tag{2.79}$$

Although this was introduced for rotations about a fixed axis, it also holds true for rotations about a **fixed point.** In this case, the angular velocity vector $\boldsymbol{\omega}(t)$ can change during the motion. We see that there is a linear relationship between the particle's velocity $\mathbf{v}_k$ and its position $\mathbf{r}_k$, which we should be able to represent using a tensor, which we shall denote by $\mathcal{R}[\boldsymbol{\omega}]$. In Cartesian coordinates, (or indeed in any system employing an orthonormal basis), the components of $\mathcal{R}[\boldsymbol{\omega}]$ may be found readily:

$$\mathbf{v}_k^{\mathrm{c}} = \boldsymbol{\omega}^{\mathrm{c}} \times \mathbf{r}_k^{\mathrm{c}} = \begin{pmatrix} \omega_y z_k - \omega_z y_k \\ \omega_z x_k - \omega_x z_k \\ \omega_x y_k - \omega_y x_k \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}}_{\mathcal{R}[\boldsymbol{\omega}]^{\mathrm{c}}} \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} \tag{2.80}$$

From this, we see that $\mathcal{R}[\boldsymbol{\omega}]$ is an example of an antisymmetric tensor. Since the cross product is invariant under a rotation of the coordinate system, for any orthogonal matrix $\mathbf{A}$ with determinant $+1$, we know that

$$\mathbf{u} \times \mathbf{v} = \mathbf{w} \Rightarrow (\mathbf{A}\mathbf{u}) \times (\mathbf{A}\mathbf{v}) = (\mathbf{A}\mathbf{w}) \tag{2.81}$$

In terms of the tensor notation, this means that

$$\mathbf{A}\mathbf{w} = \mathcal{R}[\mathbf{A}\mathbf{u}](\mathbf{A}\mathbf{v}) \tag{2.82}$$

or

$$\mathbf{w} = \mathbf{A}^{-1}\mathcal{R}[\mathbf{A}\mathbf{u}](\mathbf{A}\mathbf{v}) \tag{2.83}$$

This shows that

$$\mathcal{R}[\mathbf{u}] = \mathbf{A}^{-1}\mathcal{R}[\mathbf{A}\mathbf{u}]\mathbf{A} \tag{2.84}$$

and similarly,

$$\mathcal{R}[\mathbf{A}\mathbf{u}] = \mathbf{A}\mathcal{R}[\mathbf{u}]\mathbf{A}^{-1} \tag{2.85}$$

2.6.5 Kinetic Energy of Rotation of a Rigid System

It is also easy to calculate the kinetic energy T of the rotating rigid system of particles. It is

$$\begin{aligned} T &= \sum_k \frac{1}{2} m_k v_k^2 = \sum_k \frac{1}{2} m_k (\boldsymbol{\omega} \times \mathbf{r}_k) \cdot (\boldsymbol{\omega} \times \mathbf{r}_k) \\ &= \sum_k \frac{1}{2} m_k \boldsymbol{\omega} \cdot (\mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k)) = \frac{1}{2}\boldsymbol{\omega} \cdot \left(\sum_k m_k \mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k) \right) \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \end{aligned} \tag{2.86}$$

Note that we have cyclically permuted the order of factors in the scalar triple product. Since $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, we can write

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{t}}\mathbf{I}\boldsymbol{\omega} \tag{2.87}$$

which is a quadratic form in $\boldsymbol{\omega}$. This is the analogue of $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\mathbf{v}^{\mathrm{t}}\mathbf{v}\right)$ for linear motion. Using equation (2.57) for the inertia tensor,

$$T = \frac{1}{2}\sum_k m_k \left(r_k^2\omega^2 - \left(\mathbf{r}_k \cdot \boldsymbol{\omega}\right)^2\right) \tag{2.88}$$

$$= \frac{1}{2}\sum_k m_k \left(r_k^2\omega^2 - \left(r_k\omega\cos\theta_k\right)^2\right) \tag{2.89}$$

$$= \frac{1}{2}\sum_k m_k r_k^2\omega^2\sin^2\theta_k = \frac{1}{2}I\omega^2 \tag{2.90}$$

where θ_k is the angle between $\mathbf{r}_k$ and $\boldsymbol{\omega}$ and I is the usual moment of inertia about the rotation axis defined in terms of the perpendicular distances $r_k\sin\theta_k$ of the masses from this axis.

2.6.6 Principal Axes

We have seen that the relationship between the angular velocity and the angular momentum is linear, and is specified by a symmetric matrix $\mathbf{I}$ called the inertia tensor. In general, this means that the angular velocity vector $\boldsymbol{\omega}$ and the angular momentum vector $\mathbf{L}$ are not parallel to each other. As a consequence, it usually requires a torque to maintain a system rotating about a fixed axis, for which $\boldsymbol{\omega}$ is a constant but $\mathbf{L}$ is not.

From our knowledge of linear algebra, we know that a symmetric matrix $\mathbf{I}$ always has eigenvectors which can be chosen to be orthogonal to each other and which form a basis. Recall that a non-zero vector $\mathbf{u}$ is said to be an eigenvector of $\mathbf{I}$ belonging to eigenvalue λ if

$$\mathbf{I}\mathbf{u} = \lambda\mathbf{u} \tag{2.91}$$

If the angular velocity $\boldsymbol{\omega}$ is chosen so as to be along an eigenvector of $\mathbf{I}$, the angular momentum $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ is **parallel** to the angular velocity and the constant of proportionality is the eigenvalue.

The three mutually orthogonal eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ of the inertia tensor define the three **principal axes** of the system of particles. The eigenvalues are called the principal moments of inertia, I_1, I_2 and I_3. It turns out that the principal axes of a system are fixed in the frame of the system, so that as the system rotates in space, so do the directions of the principal axes.

For symmetrical objects such as boxes, cones, cylinders etc., it is usually easy to identify the principal axes, but they always exist and may be found by diagonalizing the inertia tensor. When working with a coordinate system fixed in the rotating body, it is usually most convenient to use the principal axes as the body axes since the matrix relationship reduces to the simple relationships $L_x = I_1\omega_x$, $L_y = I_2\omega_y$, $L_z = I_3\omega_z$ which hold true **only** in this special coordinate system. In coordinates fixed relative to space, we have the **tensor** relationship $\mathbf{L}(t) = \mathbf{I}(t)\,\boldsymbol{\omega}(t)$ where, in general, everything depends on time.

2.7 Dynamics of Rotating Rigid Bodies

It is instructive to compare the dynamics of translational motion with that of rotational motion for a rigid body. For translational motion, Newton's second law states that the total external force is equal to the rate of change of linear momentum $\mathbf{p}$,

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{2.92}$$

The linear momentum is related to the velocity $\mathbf{v}$ by the relationship

$$\mathbf{p} = m\mathbf{v}, \tag{2.93}$$

where the mass m is a scalar constant. Given the velocity, one can calculate the position $\mathbf{r}$ by integration with respect to time, or equivalently by solving the differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}. \tag{2.94}$$

In rotational dynamics, several important differences make the passage analogous to that from force to position more complicated. If we consider rotation about the centre of mass, the rotational analogue of Newton's second law is

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{2.95}$$

where $\mathbf{N}$ is the total external torque and $\mathbf{L}$ is the angular momentum. The relationship between $\mathbf{L}$ and the angular velocity $\boldsymbol{\omega}$ is still linear but is given by a **tensor** relationship

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \tag{2.96}$$

Besides the fact that unlike $\mathbf{v}$ and $\mathbf{p}$, the rotational quantities $\boldsymbol{\omega}$ and $\mathbf{L}$ are not necessarily parallel to each other, another complication is that the components of $\mathbf{I}$ (unlike m) are only constant if one works in a coordinate system fixed with respect to the rotating body. In the inertial coordinate system (fixed in space), the components of $\mathbf{I}$ change as the body rotates. We shall see that a convenient way to deal with this problem is to use a coordinate system fixed with respect to the rotating body.

Once $\boldsymbol{\omega}$ has been calculated, it is still necessary to find the orientation of the body and to determine how this orientation changes in time. Unlike position, which can be represented by a **vector** $\mathbf{r}(t)$, orientation about a point cannot be represented by a vector since in general the operations of applying finite rotations do not commute. Instead, we shall see that a suitable way of specifying the orientation is in terms of a 3×3 **rotation matrix** $\mathbf{A}(t)$ which has special properties. In particular, we shall derive the (non-linear) differential equations which relate the evolution of $\mathbf{A}(t)$ to the angular velocity $\boldsymbol{\omega}(t)$. Solving these differential equations leads to the orientation of the body as a function of time.

2.7.1 Matrix description of Rotations

For a system of particles we wrote the position vector of the i'th particle at time t as $\mathbf{r}_i(t)$. If the system is undergoing rotations about an origin so that the distance of particle i from the origin does not change with time, we can write

$$\mathbf{r}_i(t) = \mathbf{A}(t)\,\mathbf{r}_i(0) \tag{2.97}$$

where $\mathbf{A}(t)$ is a 3×3 matrix which is the **same** for all the particles. Clearly we must have $\mathbf{A}(0) = \mathbf{1}$, the identity matrix. Since for each i, we want $\|\mathbf{r}_i(t)\| = \|\mathbf{r}_i(0)\|$ for all time, the matrix $\mathbf{A}$ must be orthogonal, i.e., $\mathbf{A}^{\mathrm{t}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{t}} = \mathbf{1}$. Such a matrix preserves lengths and angles and is said to define an **isometry.** The determinant of an orthogonal matrix must be either $+1$ or -1 (since $1 = \det(\mathbf{1}) = \det(\mathbf{A}^{\mathrm{t}}\mathbf{A}) = \det(\mathbf{A}^{\mathrm{t}})\det(\mathbf{A}) = \det(\mathbf{A})^2$). For a rotation, we need to restrict ourselves to matrices with $\det(\mathbf{A}) = 1$ since matrices with negative determinants involve reflections which reverse the handedness of triads of vectors. We call an orthogonal matrix with determinant 1 a **rotation matrix.**

Since rotations are represented by matrix multiplication which is not commutative, we see that the order in which a set of rotations is applied is important. The exception to this rule is when the rotations take place about a **fixed axis** in which case we can simply add together the angles of rotation.

2.7.2 Instantaneous Angular Velocity

As discussed previously, The instantaneous angular velocity vector $\boldsymbol{\omega}(t)$ is defined such that, within the interval t to $t + \mathrm{d}t$, the body rotates through an angle $\omega(t)\,\mathrm{d}t$ around an instantaneous axis of rotation pointing in the direction of $\boldsymbol{\omega}(t)$. The velocity of the i'th particle is given by (2.79) as

$$\mathbf{v}_i(t) = \boldsymbol{\omega}(t) \times \mathbf{r}_i(t) \tag{2.98}$$

which may be written in matrix form as

$$\mathbf{v}_i(t) = \mathcal{R}[\boldsymbol{\omega}(t)]\,\mathbf{r}_i(t) \equiv \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \mathbf{r}_i(t) \tag{2.99}$$

Differentiating equation (2.97) we see that

$$\mathbf{v}_i(t) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}(t)\,\mathbf{r}_i(0) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}(t)\,\{\mathbf{A}(t)\}^{-1}\,\mathbf{r}_i(t) \tag{2.100}$$

where we have used (2.97) again in the second equality to write $\mathbf{r}_i(0)$ in terms of $\mathbf{r}_i(t)$. Comparing this with (2.99), we see that

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\mathbf{A}^{-1} = \mathcal{R}[\boldsymbol{\omega}] \tag{2.101}$$

where the explicit time-dependencies have been left out for convenience. If we take $\mathbf{A}^{-1}$ to the other side, this may be regarded as a differential equation for determining $\mathbf{A}(t)$ once we are given the time-dependent angular velocity, since we have an initial value problem

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathcal{R}[\boldsymbol{\omega}]\,\mathbf{A}, \quad \mathbf{A}(0) = \mathbf{1}. \tag{2.102}$$

Example: Rotation with constant angular velocity about z

Suppose that we consider a body rotating about O with a **constant** angular velocity about z, i.e., $\boldsymbol{\omega} = (0, 0, \omega)$. The transformation matrix may be found by integrating the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -\omega a_{21} & -\omega a_{22} & -\omega a_{23} \\ \omega a_{11} & \omega a_{12} & \omega a_{13} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.103}$$

with initial conditions

$$\begin{aligned} a_{11}(0) &= a_{22}(0) = a_{33}(0) = 1 \\ a_{12}(0) &= a_{13}(0) = a_{21}(0) = a_{23}(0) = a_{31}(0) = a_{32}(0) = 0 \end{aligned}$$

The result is exactly what we would expect

$$\mathbf{A}(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.104}$$

Exercise: (for those familiar with Laplace transforms of matrices) Rotation with constant angular velocity about an arbitrary axis:

Show that if the angular velocity $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z) = \omega(n_x, n_y, n_z)$ is constant,

$$\begin{aligned} \mathbf{A}(t) &= (1 - \cos\omega t)\begin{pmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_y n_x & n_y^2 & n_y n_z \\ n_z n_x & n_z n_y & n_z^2 \end{pmatrix} + \begin{pmatrix} \cos\omega t & -n_z\sin\omega t & n_y\sin\omega t \\ n_z\sin\omega t & \cos\omega t & -n_x\sin\omega t \\ -n_y\sin\omega t & n_x\sin\omega t & \cos\omega t \end{pmatrix} && (2.105) \\ &= \mathbf{n}\mathbf{n}^{\mathrm{t}} + (\cos\omega t)\left(\mathbf{1} - \mathbf{n}\mathbf{n}^{\mathrm{t}}\right) + (\sin\omega t)\,\mathcal{R}[\mathbf{n}] && (2.106) \end{aligned}$$

and interpret this result geometrically.

2.7.3 Body Axes and Body Coordinates

Up to this point, we have been regarding our vectors as being expressed in terms of a set of Cartesian coordinates fixed in space. In considering the motion of a rigid body, it is often convenient to consider quantities referred to a Cartesian coordinate system attached to the moving body. At time t, such a Cartesian coordinate system has basis vectors which are given by the current positions of the particles in the body which were initially at $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. The matrix $\mathbf{A}$ clearly plays a role in relating the coordinate systems. If we refer to these body axes as x_b, y_b and z_b, the coordinates of the body basis vectors in the space frame at time t are

$$\hat{\mathbf{x}}_{\mathrm{b}}^{\mathrm{s}}(t) = \mathbf{A}(t)\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \hat{\mathbf{y}}_{\mathrm{b}}^{\mathrm{s}}(t) = \mathbf{A}(t)\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{and} \quad \hat{\mathbf{z}}_{\mathrm{b}}^{\mathrm{s}}(t) = \mathbf{A}(t)\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{2.107}$$

since $\mathbf{A}$ is the matrix which rotates the body from its original position to its orientation at time t. We can write this in matrix form

$$\begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\begin{pmatrix}\vdots & \vdots & \vdots\\ \hat{\mathbf{x}}_{\mathrm{b}}^{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{b}}^{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{b}}^{\mathrm{s}}\\ \vdots & \vdots & \vdots\end{pmatrix} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\mathbf{A} \tag{2.108}$$

Consider any vector $\mathbf{G}$. Its coordinates in the two frames are related by:

$$\mathbf{G} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\mathbf{G}^{\mathrm{b}} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\mathbf{G}^{\mathrm{s}} \tag{2.109}$$

Using the transformation matrix $\mathbf{A}$, we see that

$$\begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\mathbf{G}^{\mathrm{b}} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\mathbf{A}\mathbf{G}^{\mathrm{b}} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\mathbf{G}^{\mathrm{s}} \tag{2.110}$$

and so

$$\mathbf{G}^{\mathrm{b}} = \mathbf{A}^{-1}\mathbf{G}^{\mathrm{s}} \quad \text{or} \quad \mathbf{G}^{\mathrm{s}} = \mathbf{A}\mathbf{G}^{\mathrm{b}}. \tag{2.111}$$

The time derivatives of $\begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\mathbf{A}\right] = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} \tag{2.112}$$

$$= \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} \tag{2.113}$$

The time derivative of a the vector $\mathbf{G}$ is thus given by

$$\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\mathbf{G}^{\mathrm{b}}\right] \tag{2.114}$$

$$= \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\mathbf{G}^{\mathrm{b}} + \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\frac{\mathrm{d}\mathbf{G}^{\mathrm{b}}}{\mathrm{d}t} \tag{2.115}$$

$$= \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\left\{\frac{\mathrm{d}\mathbf{G}^{\mathrm{b}}}{\mathrm{d}t} + \mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\mathbf{G}^{\mathrm{b}}\right\} \tag{2.116}$$

Using equation (2.102), we can write $\mathrm{d}\mathbf{A}/\mathrm{d}t$ as $\mathcal{R}[\boldsymbol{\omega}]\,\mathbf{A}$ where the angular velocity vector should be written as $\boldsymbol{\omega}^{\mathrm{s}}$ since it is measured in the space coordinate system. Thus

$$\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\left\{\frac{\mathrm{d}\mathbf{G}^{\mathrm{b}}}{\mathrm{d}t} + \mathbf{A}^{-1}\mathcal{R}[\boldsymbol{\omega}^{\mathrm{s}}]\,\mathbf{A}\mathbf{G}^{\mathrm{b}}\right\} \tag{2.117}$$

Using (2.85), we see that

$$\mathbf{A}^{-1}\mathcal{R}[\boldsymbol{\omega}^{\mathrm{s}}]\,\mathbf{A} = \mathcal{R}\left[\mathbf{A}^{-1}\boldsymbol{\omega}^{\mathrm{s}}\right] = \mathcal{R}\left[\boldsymbol{\omega}^{\mathrm{b}}\right] \tag{2.118}$$

where $\boldsymbol{\omega}^{\mathrm{b}} = \mathbf{A}^{-1}\boldsymbol{\omega}^{\mathrm{s}}$ is the angular velocity measured in the body coordinates. By the definition of the cross product, we see that

$$\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\left\{\frac{\mathrm{d}\mathbf{G}^{\mathrm{b}}}{\mathrm{d}t} + \boldsymbol{\omega}^{\mathrm{b}} \times \mathbf{G}^{\mathrm{b}}\right\} \tag{2.119}$$

This is the expression for the time derivative of a vector in the body coordinates. Using the fact that in the space coordinates,

$$\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{s}} & \hat{\mathbf{y}}_{\mathrm{s}} & \hat{\mathbf{z}}_{\mathrm{s}}\end{pmatrix}\frac{\mathrm{d}\mathbf{G}^{\mathrm{s}}}{\mathrm{d}t} = \begin{pmatrix}\hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}}\end{pmatrix}\mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{G}^{\mathrm{s}}}{\mathrm{d}t} \tag{2.120}$$

we can also write

$$\frac{\mathrm{d}\mathbf{G}^{\mathrm{s}}}{\mathrm{d}t} = \mathbf{A}\left\{\frac{\mathrm{d}\mathbf{G}^{\mathrm{b}}}{\mathrm{d}t} + \boldsymbol{\omega}^{\mathrm{b}} \times \mathbf{G}^{\mathrm{b}}\right\}. \tag{2.121}$$

Recall that when we considered dynamics in a uniformly rotating coordinate system, such as for the surface of the earth, we obtained essentially the same result. In that situation, however, we did not distinguish between $\boldsymbol{\omega}^{\mathrm{s}}$ and $\boldsymbol{\omega}^{\mathrm{b}}$ since these were equal. For general rigid body motion, it is important to realise that it is the angular velocity with respect to the **body coordinates** which appears in the above equation.

Example Evolution of the Rotation Matrix in terms of the Angular Velocity

We previously found that the rotation matrix $\mathbf{A}(t)$ satisfies the following differential equation involving the angular velocity $\boldsymbol{\omega}^{\mathrm{s}}$ expressed in space coordinates

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathcal{R}\left[\boldsymbol{\omega}^{\mathrm{s}}\right]\mathbf{A}, \quad \mathbf{A}(0) = \mathbf{1} \tag{2.122}$$

This can be expressed in terms of $\boldsymbol{\omega}^{\mathrm{b}}$ by premultiplying by $\mathbf{A}^{-1}$ to give

$$\mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{A}^{-1}\mathcal{R}\left[\boldsymbol{\omega}^{\mathrm{s}}\right]\mathbf{A} = \mathcal{R}\left[\mathbf{A}^{-1}\boldsymbol{\omega}^{\mathrm{s}}\right] = \mathcal{R}\left[\boldsymbol{\omega}^{\mathrm{b}}\right] \tag{2.123}$$

Hence the initial value problem which must be solved if we know $\boldsymbol{\omega}^{\mathrm{b}}(t)$ is

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{A}\mathcal{R}\left[\boldsymbol{\omega}^{\mathrm{b}}\right], \quad \mathbf{A}(0) = \mathbf{1} \tag{2.124}$$

which differs from the previous relationship only in the order of the factors on the right-hand side.

2.7.4 The Euler Equations of Motion

Consider applying equation (2.119) to the angular momentum $\mathbf{L}$ of a rigid body:

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \begin{pmatrix} \hat{\mathbf{x}}_{\mathrm{b}} & \hat{\mathbf{y}}_{\mathrm{b}} & \hat{\mathbf{z}}_{\mathrm{b}} \end{pmatrix}\left\{\frac{\mathrm{d}\mathbf{L}^{\mathrm{b}}}{\mathrm{d}t} + \boldsymbol{\omega}^{\mathrm{b}} \times \mathbf{L}^{\mathrm{b}}\right\} \tag{2.125}$$

where $\mathbf{N}$ is the applied torque. If we write $\mathbf{N}$ in the body coordinate system,

$$\mathbf{N}^{\mathrm{b}} = \frac{\mathrm{d}\mathbf{L}^{\mathrm{b}}}{\mathrm{d}t} + \boldsymbol{\omega}^{\mathrm{b}} \times \mathbf{L}^{\mathrm{b}} \tag{2.126}$$

which is the rotational analogue of $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$.

The angular momentum $\mathbf{L}$ is related to the angular velocity by the tensor relationship $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, which holds in every coordinate system. In the body coordinate system, we have that $\mathbf{L}^{\mathrm{b}} = \mathbf{I}^{\mathrm{b}}\boldsymbol{\omega}^{\mathrm{b}}$. The body coordinate system is particularly convenient for writing down this relationship since $\mathbf{I}^{\mathrm{b}}$ is **constant** in this frame (recall that $\mathbf{I}$ depends on the geometry of the object, and so it can only be constant in a frame which moves with the rotating body). We may choose the body coordinate system so that it is aligned with the principal axes of the body. With this choice, the inertia tensor components $\mathbf{I}^{\mathrm{b}}$ is diagonal and we can write

$$\mathbf{I}^{\mathrm{b}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{2.127}$$

so that $L_x^{\mathrm{b}} = I_1\omega_x^{\mathrm{b}}$, $L_y^{\mathrm{b}} = I_2\omega_y^{\mathrm{b}}$ and $L_z^{\mathrm{b}} = I_3\omega_z^{\mathrm{b}}$. The x component of the equation of motion (2.126) becomes

$$\begin{aligned} N_x^{\mathrm{b}} &= \frac{\mathrm{d}L_x^{\mathrm{b}}}{\mathrm{d}t} + \left(\omega_y^{\mathrm{b}}L_z^{\mathrm{b}} - \omega_z^{\mathrm{b}}L_y^{\mathrm{b}}\right) \\ &= I_1\frac{\mathrm{d}\omega_x^{\mathrm{b}}}{\mathrm{d}t} + \omega_y^{\mathrm{b}}I_3\omega_z^{\mathrm{b}} - \omega_z^{\mathrm{b}}I_2\omega_y^{\mathrm{b}} = I_1\frac{\mathrm{d}\omega_x^{\mathrm{b}}}{\mathrm{d}t} + (I_3 - I_2)\,\omega_y^{\mathrm{b}}\omega_z^{\mathrm{b}} \end{aligned} \tag{2.128}$$

Writing this together with the corresponding relations for the y and z directions yields

$$I_1 \frac{\mathrm{d}\omega_x^{\mathrm{b}}}{\mathrm{d}t} = (I_2 - I_3)\, \omega_y^{\mathrm{b}} \omega_z^{\mathrm{b}} + N_x^{\mathrm{b}} \tag{2.129}$$

$$I_2 \frac{\mathrm{d}\omega_y^{\mathrm{b}}}{\mathrm{d}t} = (I_3 - I_1)\, \omega_z^{\mathrm{b}} \omega_x^{\mathrm{b}} + N_y^{\mathrm{b}} \tag{2.130}$$

$$I_3 \frac{\mathrm{d}\omega_z^{\mathrm{b}}}{\mathrm{d}t} = (I_1 - I_2)\, \omega_x^{\mathrm{b}} \omega_y^{\mathrm{b}} + N_z^{\mathrm{b}} \tag{2.131}$$

which are called the **Euler equations of motion**. This is expressed completely in terms of quantities in body coordinates and they are analogous to the equations $m\mathbf{a} = \mathbf{F}$ for translational motion. As mentioned previously, the advantage of using body coordinates is that the inertia tensor is a constant in this frame as it follows the motion of the body. These are as set of **non-linear** equations for $\boldsymbol{\omega}_b$. They can be solved together with $\dot{\mathbf{A}} = \mathbf{A}\mathcal{R}\left[\boldsymbol{\omega}_b\right]$ numerically to give the time-dependent rotation matrix $\mathbf{A}(t)$ for the orientation of the rigid body. Due to the non-linearity, it is difficult to solve these equations analytically except in special situations.

2.7.5 Stability of Free Rotation about the Principal Axes

Free rotation refers to the situation where there are no torques acting so that $\mathbf{N} = \mathbf{0}$. This means that the angular momentum is conserved so that $\mathbf{L}$ is fixed in both magnitude and direction. For translational motion, this is analogous to motion in the absence of forces, and by Netwton's first law, the body continues in a state of rest or of motion at constant velocity. We shall see however that for free rotational motion, the angular velocity is **not** constant in general, and the rigid body can undergo rather complicated motion.

If we restrict ourselves to motion about a principal axis, for which the angular velocity and angular momentum are parallel, one might be tempted to conclude that since $\mathbf{L}$ is constant, the angular velocity $\boldsymbol{\omega}$ must also be constant. Rather surprisingly, this is only true if rotation occurs **precisely** about each principal axis. For a general rigid body, there are three different moments of inertia I_1, I_2 and I_3. If there is a very small deviation from the exact principal direction, it is found that the deviation grows rapidly if the principal direction is that associated with the **intermediate** moment of inertia although the motion is stable about the axes with the largest and smallest moments of inertia. We shall use the Euler equations to investigate the motion.

When there are no torques acting, the Euler equations of motion are

$$I_1 \frac{\mathrm{d}\omega_x^{\mathrm{b}}}{\mathrm{d}t} = (I_2 - I_3)\, \omega_y^{\mathrm{b}} \omega_z^{\mathrm{b}} \tag{2.132}$$

$$I_2 \frac{\mathrm{d}\omega_y^{\mathrm{b}}}{\mathrm{d}t} = (I_3 - I_1)\, \omega_z^{\mathrm{b}} \omega_x^{\mathrm{b}} \tag{2.133}$$

$$I_3 \frac{\mathrm{d}\omega_z^{\mathrm{b}}}{\mathrm{d}t} = (I_1 - I_2)\, \omega_x^{\mathrm{b}} \omega_y^{\mathrm{b}} \tag{2.134}$$

Being non-linear, these equations are difficult to deal with analytically. However, we wish to consider the case where the rotation is started off along one principal axis, say the x axis with angular velocity ω_0. For small times, we expect that the angular velocity will be close to this initial condition, so that we can write

$$\boldsymbol{\omega}^{\mathrm{b}}(t) = \begin{pmatrix} \omega_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon_x(t) \\ \varepsilon_y(t) \\ \varepsilon_z(t) \end{pmatrix} \tag{2.135}$$

where ε_x, ε_y and ε_z are to be thought of as perturbations which are small in comparison with ω_0. If we substitute this supposed solution into the Euler equations, we obtain

$$I_1 \frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t} = (I_2 - I_3)\, \varepsilon_y \varepsilon_z \tag{2.136}$$

$$I_2 \frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t} = (I_3 - I_1)\, \varepsilon_z (\omega_0 + \varepsilon_x) \tag{2.137}$$

$$I_3 \frac{\mathrm{d}\varepsilon_z}{\mathrm{d}t} = (I_1 - I_2)\, (\omega_0 + \varepsilon_x)\, \varepsilon_y \tag{2.138}$$

we now proceed to ignore terms on the right hand side which are of higher than the first order in the small perturbations. This procedure is often referred to as **linearising** the equations of motion.

$$I_1 \frac{d\varepsilon_x}{dt} = 0 \tag{2.139}$$

$$I_2 \frac{d\varepsilon_y}{dt} = (I_3 - I_1)\,\omega_0 \varepsilon_z \tag{2.140}$$

$$I_3 \frac{d\varepsilon_z}{dt} = (I_1 - I_2)\,\omega_0 \varepsilon_y \tag{2.141}$$

Since these equations are now linear, we seek solutions of the form $\varepsilon_y(t) = \varepsilon_{y0} \exp(st)$ and $\varepsilon_z(t) = \varepsilon_{z0} \exp(st)$ for some s. Substituting these into the last two equations, we find

$$I_2 s \varepsilon_{y0} = (I_3 - I_1)\,\omega_0 \varepsilon_{z0} \tag{2.142}$$

$$I_3 s \varepsilon_{z0} = (I_1 - I_2)\,\omega_0 \varepsilon_{y0} \tag{2.143}$$

since the exponentials cancel. These form a homogeneous set of linear equations

$$\begin{pmatrix} sI_2 & (I_1 - I_3)\,\omega_0 \\ (I_2 - I_1)\,\omega_0 & sI_3 \end{pmatrix} \begin{pmatrix} \varepsilon_{y0} \\ \varepsilon_{z0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.144}$$

This system of equations only has the zero solution unless the determinant of the matrix is equal to zero. For non-trivial solutions, we see that

$$s^2 I_2 I_3 - (I_1 - I_3)(I_2 - I_1)\,\omega_0^2 = 0 \tag{2.145}$$

or

$$s = \pm\omega_0 \sqrt{\frac{(I_1 - I_3)(I_2 - I_1)}{I_2 I_3}}. \tag{2.146}$$

We see that if I_1 is either the largest or the smallest moment of inertia, then s is purely imaginary and we have $s = \pm i\lambda$ for some λ. Under these conditions, the perturbation is of the form $\begin{pmatrix} \varepsilon_{y0}^{\pm} \\ \varepsilon_{z0}^{\pm} \end{pmatrix} \exp(\pm i\lambda t)$ which oscillates and does not grow with time. On the other hand, if I_1 is the intermediate moment of inertia, i.e., that $I_2 > I_1 > I_3$ or $I_3 > I_1 > I_2$, then there is one positive value of s and one negative value of s and the perturbation is of the form $\begin{pmatrix} \varepsilon_{y0}^{\pm} \\ \varepsilon_{z0}^{\pm} \end{pmatrix} \exp(\pm\lambda t)$ for some real $\lambda > 0$. The growing exponential means that ω_y and ω_z will become large and the object no longer rotates about the $\hat{\mathbf{x}}_b$ axis. At some stage, the perturbation theory breaks down, and we need to consider the full numerical solution to discover what is actually going to happen. In Figure 2.3, the components of $\boldsymbol{\omega}^b$ are shown for a body initially rotating nearly about the x axis. Initially, the growth in ω_y^b and ω_z^b is approximately exponential and ω_x^b remains essentially constant, in agreement with the perturbation theory solution, but once the deviations are large, we see that the exponential growth breaks down and (in this specific case) the sign of ω_x^b flips periodically. Thus a rigid body set spinning about the intermediate principal axis tumbles in space as it spins. Throughout this motion, the angular momentum vector $\mathbf{L}$ remains strictly constant in magnitude and direction (with respect to space).

2.8 The Foucault Gyrocompass

Although the equation

$$\frac{d\mathbf{G}}{dt} = \begin{pmatrix} \hat{\mathbf{x}}_b & \hat{\mathbf{y}}_b & \hat{\mathbf{z}}_b \end{pmatrix} \left\{ \frac{d\mathbf{G}^b}{dt} + \boldsymbol{\omega}^b \times \mathbf{G}^b \right\} \tag{2.147}$$

was derived in terms of a coordinate system fixed in a rigid body, all that the rigid body is actually needed for is to define a right-handed Cartesian coordinate system with basis vectors $\{\hat{\mathbf{x}}_b, \hat{\mathbf{y}}_b, \hat{\mathbf{z}}_b\}$ which may rotate

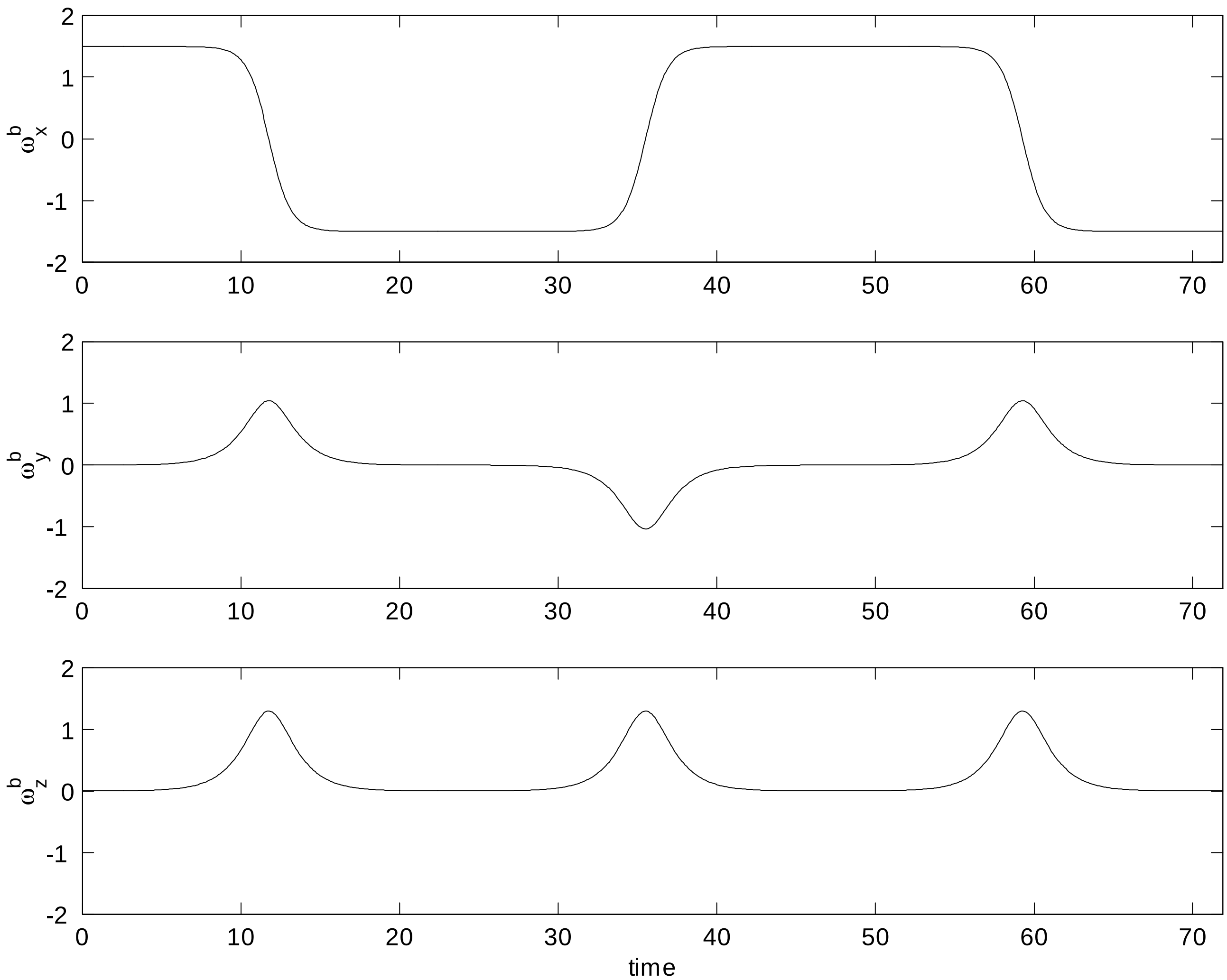

Figure 2.3 Time evolution of $\boldsymbol{\omega}^{\mathrm{b}}$ for a body initially rotating nearly about the x axis. The principal moments of inertia are $I_1 = 10$, $I_2 = 3.25$ and $I_3 = 11.25$.

in an arbitrary way, but whose origin is stationary. The vector $\boldsymbol{\omega}$ is then the angular velocity with which this coordinate system moves relative to an inertial system.

As an illustration of the use of this equation, we consider a Foucault gyrocompass, which is a rapidly spinning flywheel whose centre is fixed, and whose axle is constrained to lie in a horizontal plane. The direction which the axle points to is however free to rotate within this horizontal plane. We shall show that such a system can be used to find the direction of true north (*not* magnetic north) on the rotating earth.

Suppose that the system is at latitude λ in the northern hemisphere, and that the axle initially makes an angle of θ to true north. Figure 2.4 shows the geometry of the situation. The horizontal plane is marked with the cardinal points (North, South, East and West). The angular momentum vector of the earth, $\boldsymbol{\omega}_e$, lies in the vertical north-south plane, and makes an angle of λ with the horizontal. The axes of the right-handed Cartesian coordinate system we choose to use are denoted x', y',z', where y' points along the axle and z' points up. This coordinate system is chosen so that the inertia tensor is *diagonal*. The moment of inertia I_2 (around the y' axis) is larger than the common moment of inertia $I_1 = I_3 = I$ about the x' or z' axes. The flywheel spins with angular velocity $\mathbf{\Omega}$ about the y' axis, but this is *not* the total angular velocity of the system as seen in the inertial frame, since the angle θ can vary, and the rotation of the earth also contributes to the total angular velocity. Note also that the system x', y', z' are *not* the body coordinates of the flywheel since the body coordinate system would spin with the wheel, while we are assuming that z' always keeps pointing upwards, although y' does follow the motion of the axle.

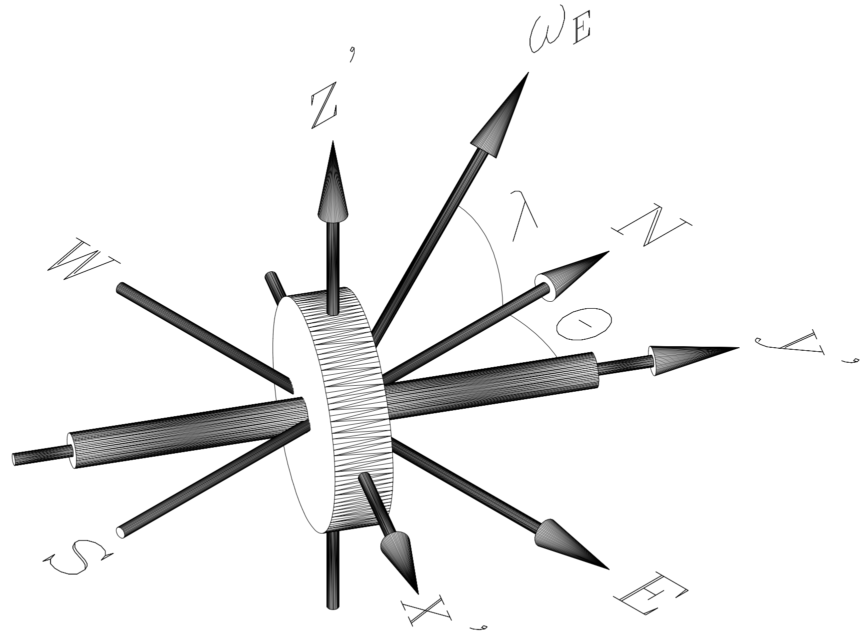

Figure 2.4 The Foucault gyrocompass

First we wish to compute the angular velocity of the (x', y', z') coordinate system relative to the inertial frame. If the angle θ increases, this corresponds to an angular velocity in the negative z' direction. Since the coordinate system is also rotated by the earth, we find that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_e - \dot{\theta}\hat{\mathbf{z}}' \tag{2.148}$$

where $\hat{\mathbf{z}}'$ is the unit vector pointing upwards. Using simple geometry, we can compute the components of $\boldsymbol{\omega}_e$ along the primed coordinate system. We find that

$$\boldsymbol{\omega}_e = -\omega_e \cos\lambda \sin\theta \hat{\mathbf{x}}' + \omega_e \cos\lambda \cos\theta \hat{\mathbf{y}}' + \omega_e \sin\lambda \hat{\mathbf{z}}' \tag{2.149}$$

Thus the components of $\boldsymbol{\omega}$ in the primed system are

$$\boldsymbol{\omega}' = \begin{pmatrix} -\omega_e \cos\lambda \sin\theta \\ \omega_e \cos\lambda \cos\theta \\ \omega_e \sin\lambda - \dot{\theta} \end{pmatrix} \tag{2.150}$$

The vector whose time-derivative we wish to compute is $\mathbf{L}$, the total angular momentum of the flywheel. The total angular velocity of the flywheel is the sum of $\boldsymbol{\omega}'$ and $\Omega\hat{\mathbf{y}}'$. Using the values of the principle moments of inertia, we see that the components of $\mathbf{L}$ in the primed coordinate system are

$$\mathbf{L}' = \begin{pmatrix} -I\omega_e \cos\lambda \sin\theta \\ I_2\left(\omega_e \cos\lambda \cos\theta + \Omega\right) \\ I\left(\omega_e \sin\lambda - \dot{\theta}\right) \end{pmatrix} \tag{2.151}$$

Substituting into (2.147), we find

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \begin{pmatrix} \hat{\mathbf{x}}' & \hat{\mathbf{y}}' & \hat{\mathbf{z}}' \end{pmatrix} \left\{ \frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t} + \boldsymbol{\omega}' \times \mathbf{L}' \right\} \tag{2.152}$$

$$= \begin{pmatrix} \hat{\mathbf{x}}' & \hat{\mathbf{y}}' & \hat{\mathbf{z}}' \end{pmatrix} \left\{ \begin{pmatrix} -I\omega_e \cos\lambda \cos\theta\, \dot{\theta} \\ I_2\left(-\omega_e \cos\lambda \sin\theta\, \dot{\theta} + \dot{\Omega}\right) \\ -I\ddot{\theta} \end{pmatrix} + \begin{pmatrix} -\omega_e \cos\lambda \sin\theta \\ \omega_e \cos\lambda \cos\theta \\ \omega_e \sin\lambda - \dot{\theta} \end{pmatrix} \times \begin{pmatrix} -I\omega_e \cos\lambda \sin\theta \\ I_2\left(\omega_e \cos\lambda \cos\theta + \Omega\right) \\ I\left(\omega_e \sin\lambda - \dot{\theta}\right) \end{pmatrix} \right\} \tag{2.153}$$

The time derivative of $\mathbf{L}$ is equal to the external torque acting on the flywheel. Since the only torque is that due to the vertical forces at the bearings which are used to keep the axle in the horizontal plane, we see that $\mathbf{N}$ must lie in the $\hat{\mathbf{x}}'$ direction. In particular, both the y' and z' components of $\mathbf{N}$ vanish.

Looking at the z' component of $\dot{\mathbf{L}}$ and equating this to zero, we find:

$$0 = -I\ddot{\theta} - \omega_e\left(\cos\lambda \sin\theta\right) I_2\left(\omega_e \cos\lambda \cos\theta + \Omega\right) + \omega_e^2\left(\cos^2\lambda \cos\theta\right) I \sin\theta \tag{2.154}$$

Since $\Omega \gg \omega_e$, we may neglect the terms involving ω_e^2 to obtain

$$\ddot{\theta} = -\left(\frac{I_2 \Omega \omega_e \cos\lambda}{I}\right) \sin\theta \tag{2.155}$$

This equation for θ shows that it behaves just like a simple pendulum with equilibrium position at $\theta = 0$, i.e., with the axle pointing due north. Thus the axle will oscillate about true north. With a small amount of damping the axle will ultimately point northwards. The period of the oscillations (in the $\sin\theta = \theta$ approximation) is

$$T = 2\pi\sqrt{\frac{I}{I_2 \Omega \omega_e \cos\lambda}} \tag{2.156}$$

Since ω_e is rather small (approx $7.3 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$), the flywheel rotation rate Ω must be large in order to give a reasonably small T. For example, if $I/\left(I_2 \cos\theta\right) \approx 1$, we need a rotation rate of 100 revolutions per second (6000 rpm) to give $T \approx 29\,\mathrm{s}$. This will allow us to find the direction of north in a few minutes.

2.9 The Euler Angles

Although a rotation matrix is of size 3×3 and thus contains nine real numbers, the orthogonality conditions provide six constraints which mean that only three numbers can be specified independently. One way of specifying a rotation is to give the polar coordinates (θ, ϕ) of the rotation axis and the angle of rotation about this axis. An alternative method is to use the Euler angles which provide a way of specifying an arbitrary rotation matrix in terms of three rotations applied in succession. The three operations are

1. Rotation about the $\hat{\mathbf{z}}_\mathrm{s}$ axis through angle ψ, followed by

2. Rotation about the $\hat{\mathbf{x}}_{\mathrm{s}}$ axis through angle θ, followed by
3. Rotation about the $\hat{\mathbf{z}}_{\mathrm{s}}$ axis through angle ϕ,

so that the overall rotation matrix is

$$\begin{aligned}
\mathbf{A} &= \mathbf{R}(\hat{\mathbf{z}}_{\mathrm{s}},\phi)\,\mathbf{R}(\hat{\mathbf{x}}_{\mathrm{s}},\theta)\,\mathbf{R}(\hat{\mathbf{z}}_{\mathrm{s}},\psi) && (2.157)\\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} && (2.158)\\
&= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix} && (2.159)
\end{aligned}$$

Let us relabel these three matrices as $\mathbf{B} = \mathbf{R}(\hat{\mathbf{z}}_{\mathrm{s}},\phi)$, $\mathbf{C} = \mathbf{R}(\hat{\mathbf{x}}_{\mathrm{s}},\theta)$ and $\mathbf{D} = \mathbf{R}(\hat{\mathbf{z}}_{\mathrm{s}},\psi)$ so that $\mathbf{A} = \mathbf{BCD}$. Now let us suppose that each of the Euler angles ψ, θ and ϕ are functions of time. We wish to find the angular velocity of the body in body coordinates. This is given by (2.124) as

$$\begin{aligned}
\mathcal{R}\left[\boldsymbol{\omega}^{\mathrm{b}}\right] &= \mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{D}^{-1}\mathbf{C}^{-1}\mathbf{B}^{-1}\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\mathbf{CD} + \mathbf{B}\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t}\mathbf{D} + \mathbf{BC}\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\right) && (2.160)\\
&= \mathbf{D}^{-1}\mathbf{C}^{-1}\left(\mathbf{B}^{-1}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)\mathbf{CD} + \mathbf{D}^{-1}\left(\mathbf{C}^{-1}\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t}\right)\mathbf{D} + \left(\mathbf{D}^{-1}\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\right) && (2.161)\\
&= \dot{\phi}\mathbf{D}^{-1}\mathbf{C}^{-1}\mathcal{R}\left[\hat{\mathbf{z}}_{\mathrm{s}}\right]\mathbf{CD} + \dot{\theta}\mathbf{D}^{-1}\mathcal{R}\left[\hat{\mathbf{x}}_{\mathrm{s}}\right]\mathbf{D} + \dot{\psi}\mathcal{R}\left[\hat{\mathbf{z}}_{\mathrm{s}}\right] && (2.162)\\
&= \dot{\phi}\mathcal{R}\left[\mathbf{D}^{-1}\mathbf{C}^{-1}\hat{\mathbf{z}}_{\mathrm{s}}\right] + \dot{\theta}\mathcal{R}\left[\mathbf{D}^{-1}\hat{\mathbf{x}}_{\mathrm{s}}\right] + \dot{\psi}\mathcal{R}\left[\hat{\mathbf{z}}_{\mathrm{s}}\right] && (2.163)
\end{aligned}$$

where we have used (2.85) in the last equality. Considering the elements of the matrices, we see that

$$\begin{aligned}
\boldsymbol{\omega}_{\mathrm{b}} &= \dot{\phi}\mathbf{D}^{-1}\mathbf{C}^{-1}\hat{\mathbf{z}}_{\mathrm{s}} + \dot{\theta}\mathbf{D}^{-1}\hat{\mathbf{x}}_{\mathrm{s}} + \dot{\psi}\hat{\mathbf{z}}_{\mathrm{s}}\\
&= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}\\
&= \begin{pmatrix} (\sin\psi\sin\theta)\,\dot{\phi} + (\cos\psi)\,\dot{\theta} \\ (\cos\psi\sin\theta)\,\dot{\phi} - (\sin\psi)\,\dot{\theta} \\ (\cos\theta)\,\dot{\phi} + \dot{\psi} \end{pmatrix}
\end{aligned}$$

Exercise: Show that the angular velocity in space coordinates is given in terms of the Euler angles by

$$\boldsymbol{\omega}_{\mathrm{s}} = \begin{pmatrix} (\cos\phi)\,\dot{\theta} + (\sin\phi\sin\theta)\,\dot{\psi} \\ (\sin\phi)\,\dot{\theta} - (\cos\phi\sin\theta)\,\dot{\psi} \\ \dot{\phi} + (\cos\theta)\,\dot{\psi} \end{pmatrix}$$

A more traditional way of defining the Euler angles is through the use of a succession of rotations about the **body** axes rather than the **space** axes. Remarkably, the sequence of rotations given above is equivalent to the following sequence. Suppose that the original basis vectors when the body and space axes are coincident are $\hat{\mathbf{x}}_{\mathrm{s}}$, $\hat{\mathbf{y}}_{\mathrm{s}}$ and $\hat{\mathbf{z}}_{\mathrm{s}}$. The three operations are:

1. Rotation about the $\hat{\mathbf{z}}_{\mathrm{s}}$ axis through angle ϕ. The new body basis vectors are $\hat{\mathbf{x}}_1$, $\hat{\mathbf{y}}_1$ and $\hat{\mathbf{z}}_1$,
2. Rotation about the $\hat{\mathbf{x}}_1$ axis through angle θ. The new body basis vectors are $\hat{\mathbf{x}}_2$, $\hat{\mathbf{y}}_2$ and $\hat{\mathbf{z}}_2$,
3. Rotation about the $\hat{\mathbf{z}}_2$ axis through angle ψ. The final body basis vectors are $\hat{\mathbf{x}}_{\mathrm{b}}$, $\hat{\mathbf{y}}_{\mathrm{b}}$ and $\hat{\mathbf{z}}_{\mathrm{b}}$.

These operations are shown in Figure. Note that the sequence of angles is ϕ, θ, ψ rather than ψ, θ, ϕ when we considered rotations about space axes.In order to show the equivalence, we need the result

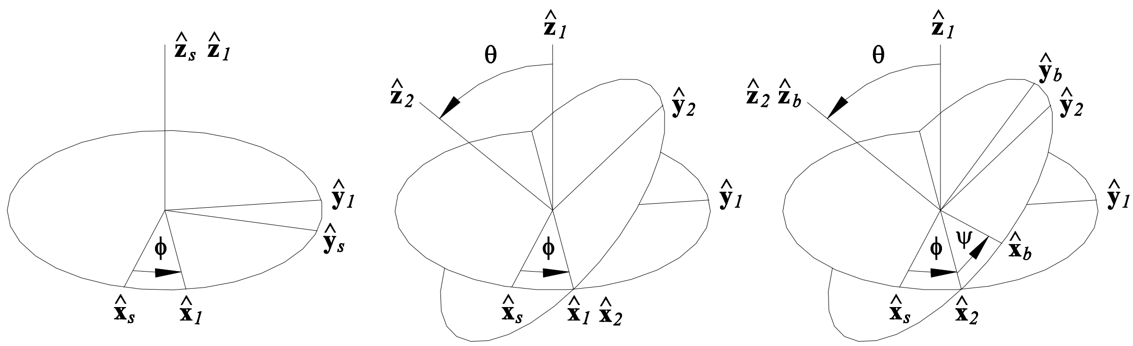

Figure 2.5 Sequence of operations in the traditional definition of the Euler angles. First a rotation of ϕ about the space axis $\hat{\mathbf{z}}_s$, second a rotation of θ about the body axis $\hat{\mathbf{x}}_1$ and third, a rotation of ψ about the body axis $\mathbf{z}_2$.

$$\hat{\mathbf{v}}' = \mathbf{A}\hat{\mathbf{v}} \quad \Rightarrow \quad \mathbf{R}\left(\hat{\mathbf{v}}', \theta\right) = \mathbf{A}\mathbf{R}\left(\hat{\mathbf{v}}, \theta\right)\mathbf{A}^{-1} \tag{2.164}$$

for any rotation matrix $\mathbf{A}$ and where $\mathbf{R}\left(\hat{\mathbf{v}}, \theta\right)$ denotes the rotation matrix for a rotation about the vector $\hat{\mathbf{v}}$ through angle θ. This simply asserts that we can rotate a vector $\mathbf{u}$ around $\mathbf{A}\hat{\mathbf{v}}$ through angle θ by first applying $\mathbf{A}^{-1}$ to both vectors, rotating $\mathbf{A}^{-1}\mathbf{u}$ around $\hat{\mathbf{v}}$ through θ and then applying $\mathbf{A}$ to the result.

The overall rotation matrix for the sequence of three rotations about the body axes is

$$\mathbf{r}_{\text{final}} = \mathbf{R}\left(\hat{\mathbf{z}}_2, \psi\right)\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_s, \phi\right)\mathbf{r}_{\text{initial}} \tag{2.165}$$

Since $\hat{\mathbf{x}}_1$ is the $\hat{\mathbf{x}}_\text{s}$ basis vector rotated through the first transformation, $\hat{\mathbf{x}}_1 = \mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\hat{\mathbf{x}}_\text{s}$. Since $\hat{\mathbf{z}}_2$ is $\hat{\mathbf{z}}_\text{s}$ rotated through the first two transformations, $\hat{\mathbf{z}}_2 = \mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\hat{\mathbf{z}}_\text{s}$. Using the result (2.164) with $\mathbf{A} = \mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)$, $\hat{\mathbf{v}}' = \hat{\mathbf{z}}_2$ and $\hat{\mathbf{v}} = \hat{\mathbf{z}}$, we obtain

$$\begin{aligned} \mathbf{R}\left(\hat{\mathbf{z}}_2, \psi\right) &= \left\{\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\right\}\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \psi\right)\left\{\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\right\}^{-1} \\ &= \mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \psi\right)\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)^{-1} \end{aligned} \tag{2.166}$$

Substituting into the expression for $\mathbf{r}_{\text{final}}$ yields

$$\mathbf{r}_{\text{final}} = \left\{\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_s, \psi\right)\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)^{-1}\right\}\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\mathbf{r}_{\text{initial}} \tag{2.167}$$

$$= \mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \psi + \phi\right)\mathbf{r}_{\text{initial}} \tag{2.168}$$

where we have combined $\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \psi\right)$ and $\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)$ since they are rotations about the same axis. Using the result (2.164) again with $\mathbf{A} = \mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)$, $\hat{\mathbf{v}}' = \hat{\mathbf{x}}_1$ and $\hat{\mathbf{v}} = \hat{\mathbf{x}}_\text{s}$, we find

$$\mathbf{R}\left(\hat{\mathbf{x}}_1, \theta\right) = \mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\mathbf{R}\left(\hat{\mathbf{x}}_\text{s}, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)^{-1} \tag{2.169}$$

Substituting into the expression for $\mathbf{r}_{\text{final}}$ yields

$$\mathbf{r}_{\text{final}} = \left\{\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\mathbf{R}\left(\hat{\mathbf{x}}_\text{s}, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)^{-1}\right\}\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \psi + \phi\right)\mathbf{r}_{\text{initial}} \tag{2.170}$$

$$= \mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \phi\right)\mathbf{R}\left(\hat{\mathbf{x}}_\text{s}, \theta\right)\mathbf{R}\left(\hat{\mathbf{z}}_\text{s}, \psi\right)\mathbf{r}_{\text{initial}} \tag{2.171}$$

which is the sequence originally stated in (2.157) in terms of the space axes.