Chapter 1 Dirac delta functions and distributions

1.1 Motivation

In many applications in physics, we would like to be able to handle quantities such as

1. The density $\rho(\mathbf{r})$ of a point mass at $\mathbf{r}_0$,
2. The probability density function of a discrete distribution,
3. The force $F(t)$ associated with an "instantaneous" collision,
4. The derivative of a function with a step discontinuity.

We may regard each of these as the result of a limiting process in which a quantity gets more and more concentrated at a point in space or time. Consider the sequence of functions

$$f_n(t) = \begin{cases} n & \text{for } |t| < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}. \tag{1.1}$$

These functions get narrower and taller as $n \to \infty$. The area in any interval enclosing the origin tends to one as n becomes large.

These functions do not tend pointwise to any limit as $n \to \infty$ but physically it is useful to think of "point" sources such as point charges, point masses etc. We would like to invent a "limit" for this sequence of functions which should have certain properties. It is called the (Dirac) **delta function** or the **unit impulse** and is denoted by $\delta(t)$. We would like $\delta(t)$ to satisfy

$$\int_a^b \delta(t)\,\mathrm{d}t = \begin{cases} 1 & \text{if } 0 \in (a,b) \\ 0 & \text{if } 0 \notin [a,b] \end{cases}. \tag{1.2}$$

Given any function $\phi(t)$ continuous at the origin, we have that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\phi(t)\,\mathrm{d}t = \phi(0), \tag{1.3}$$

and so we would like $\delta(t)$ to have the property that

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,\mathrm{d}t = \phi(0). \tag{1.4}$$

Many other sequences of functions may be found which also satisfy the limit (1.3) for a suitable class of functions ϕ. These provide alternative representations of the delta function.

1.2 Distributions

We wish to develop a theory which will allow us to deal with objects like delta functions in a correct way. These are not ordinary functions but are rather examples of **distributions** or **generalized functions**. The extra conceptual power and elegance of distribution theory will (hopefully) become obvious as we proceed. In particular, we shall find that any distribution can be differentiated (to give another distribution) any number of times and that distributions justify many operations which we often carry out in Physics calculations which are incorrect for "ordinary" functions.

1.2.1 Some concepts from mathematical analysis

The following is a rather brief summary of a few definitions and results which will be useful later. They should also serve to emphasize that we do need to be rather careful when manipulating ordinary functions.

Suppose that $\{x_n\}$ for $n = 1, 2, ...$ is a sequence of real or complex numbers. We say that the sequence **converges to** x as $n \to \infty$ if $|x_n - x|$ can be made to be as small as we like whenever n is sufficiently large. Formally, this is written $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$, such that $n \geq N \Rightarrow |x_n - x| < \epsilon$ where the quantifiers $\forall$ and $\exists$ represent "for all" and "there exist" respectively.

A function $f(t)$ is said to have **limit** b as $x \to a$ and we write

$$\lim_{t\to a} f(t) = b, \tag{1.5}$$

provided that $|f(t) - b|$ can be made as small as we like whenever t is sufficiently close (but not equal) to a. Formally, $\forall\epsilon > 0$, $\exists\delta > 0$ such that $0 < |t - a| < \delta \Rightarrow |f(t) - b| < \epsilon$.

A function $f(t)$ is said to be **continuous** at a if

$$\lim_{t\to a} f(t) = f(a), \tag{1.6}$$

i.e., $\forall\epsilon > 0$, $\exists\delta > 0$ such that $|t - a| < \delta \Rightarrow |f(t) - f(a)| < \epsilon$.

Note that continuity is a property that a function may have **at a point.** There are functions which are continuous only at a single point and are discontinuous everywhere else. Consider for example

$$f(t) = \begin{cases} t & \text{if } t \text{ is a rational number} \\ 0 & \text{if } t \text{ is irrational.} \end{cases} \tag{1.7}$$

You should be able to show (using the definition of continuity) that this function is continuous at $t = 0$ and nowhere else.

It should be familiar result from analysis that a function $f(t)$ which is continuous on a closed and bounded set K is bounded, i.e., $\exists M \in \mathbb{R}$ such that $|f(t)| \leq M$ for all $t \in K$.

We now wish to consider **sequences of functions**. Suppose $\{f_n\}$ is a sequence of functions all defined on some set S. The functions are said to **converge pointwise** to a function f as $n \to \infty$ if for each $t \in S$,

$$f(t) = \lim_{n\to\infty} f_n(t). \tag{1.8}$$

Notice that for a **given** t, the limit is simply the limit of a **sequence of numbers**.

Examples of sequences of functions

1. Define

$$f_n(t) = \begin{cases} 0 & \text{for } t < 0 \\ t^n & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } t > 1 \end{cases}.$$

   This sequence of functions (see figure 1.1) converges pointwise to the limit function

$$f(t) = \begin{cases} 0 & \text{for } t < 1 \\ 1 & \text{for } t \geq 1 \end{cases}. \tag{1.9}$$

   Note that the limiting function is discontinuous at $t = 1$, even though every function $f_n(t)$ in the sequence is continuous at $t = 1$.

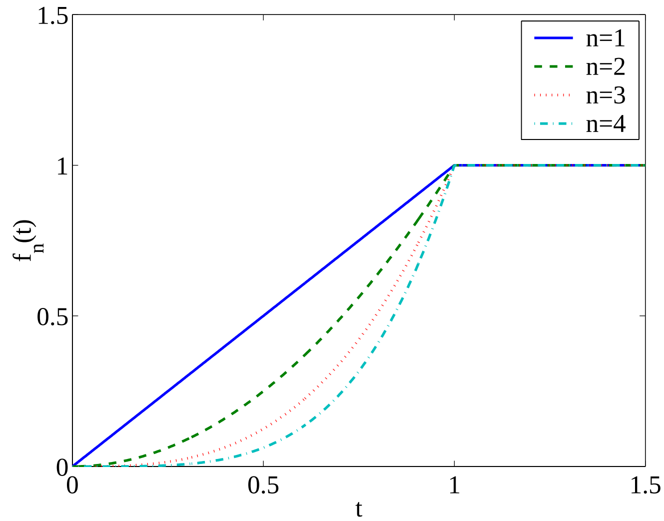

Figure 1.1 An example of a sequence of functions which converges pointwise but not uniformly to a limiting function as $n \to \infty$.

2. The sequence of functions
$$f_n(t) = \begin{cases} n & |t| < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}, \tag{1.10}$$
does **not** converge pointwise to any function since the sequence of numbers $\{f_n(0)\}$ does not converge.

3. The sequence of functions
$$f_n(t) = \begin{cases} n^2 t & \text{for } 0 \le t \le \frac{1}{n} \\ n^2\left(\frac{2}{n} - t\right) & \text{for } \frac{1}{n} < t \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases} \tag{1.11}$$
converges pointwise to the zero function. i.e., for every t, the sequence of numbers $f_n(t)$ converges to zero (see Figure 1.2). This is because given any $t > 0$, there is an integer N, such that for all $n > N$, $f_n(t) = 0$. For all other values of t, the value of $f_n(t)$ is zero for every n.

4. The sequence of functions
$$f_n(t) = \frac{1}{n}\sin(n^2 t) \tag{1.12}$$
converges pointwise to the zero function.

Example 1 shows that even when **all** the functions in a sequence are **continuous**, the pointwise limit can fail to be continuous (in this case at $t = 1$).

In example 3, each function $f_n(t)$ satisfies
$$\int_{-\infty}^{\infty} f_n(t)\,\mathrm{d}t = 1. \tag{1.13}$$

Thus the limiting value of the integral as $n \to \infty$ is one. However, the limiting function is zero and so the integral of the limit function is zero. We see in general that
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\,\mathrm{d}t \ne \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t \quad \text{even if} \quad f_n(t) \to f(t) \quad \text{pointwise.} \tag{1.14}$$

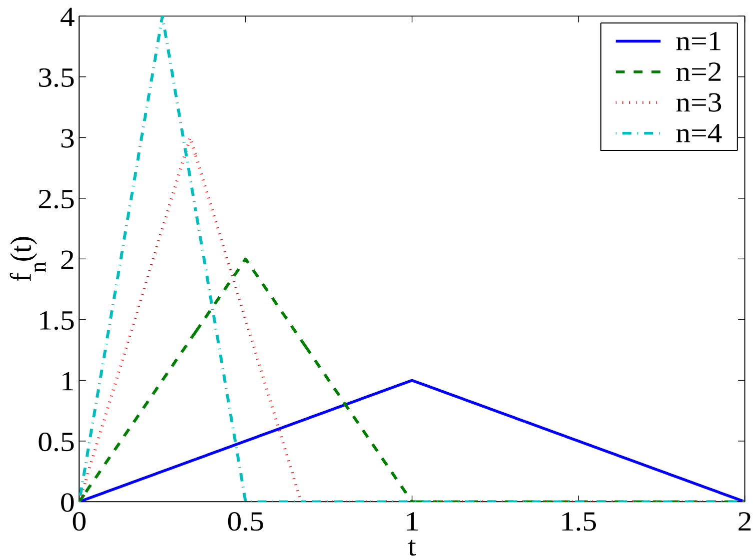

Figure 1.2 An example of a sequence of functions which converges pointwise but not uniformly to the function which is identically zero.

In example 4, the derivative sequence

$$f_n'(t) = n \cos(n^2 t) \tag{1.15}$$

does **not** converge pointwise to any function, and certainly not to $f'(t)$, the derivative of the limiting function. Thus in general

$$\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} f_n(t) \neq \frac{\mathrm{d}}{\mathrm{d}t} f(t) \quad \text{even if} \quad f_n(t) \to f(t) \quad \text{pointwise.} \tag{1.16}$$

One way of trying to improve matters is to strengthen what we mean by convergence of functions. We thus define the concept of **uniform convergence**. A sequence of functions all defined on some set S is said to **converge uniformly on** S to a function f as $n \to \infty$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall t \in S$, $|f_n(t) - f(t)| < \epsilon$ whenever $n \geq N$.

It is important to know how this is different from the concept of pointwise convergence. Translating the definition of pointwise convergence given previously into symbols reads $\forall t \in S$, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(t) - f(t)| < \epsilon$ whenever $n \geq N$. We see that for pointwise convergence, the value of N depends **both** on ϵ and on t whereas for uniform convergence, the **same** N must work for **all** $t \in S$. Notice that if a sequence of functions converges uniformly to some limit, the sequence must also converge pointwise to that limit, but the converse does not necessarily hold.

Exercise: Show that the convergence in examples 1 and 3 is **not** uniform on $\mathbb{R}$ but that convergence **is** uniform in example 4.

Another way of checking whether a pointwise-convergent sequence is uniformly convergent is to look at the **difference sequence** $d_n(t) = f_n(t) - f(t)$, which is another sequence of functions. The convergence of $f_n(t)$ to $f(t)$ is uniform on S provided that for all $t \in S$, $|d_n(t)|$ may be bounded above by a sequence which converges to zero. i.e., if we can find a sequence M_n such that $M_n \to 0$ as $n \to \infty$ and $|d_n(t)| \leq M_n$ for all $t \in S$.

If f_n converges to f **uniformly** on S, then

1. If each f_n is continuous at $a \in S$, f is also continuous at a.

2. Provided that f_n and f are integrable, the limit of the integral is equal to the integral of the limit, i.e.,
$$\lim_{n\to\infty}\int_a^x f_n(t)\,\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t, \tag{1.17}$$
where $[a, x]$ is contained in S. The convergence of the integral is uniform.

Exercise: Prove the above assertions using the definitions given.

However, example 4 shows that we still **cannot** interchange the limiting process and a derivative, even given uniform convergence. This whole situation is rather unsatisfactory, especially since uniform convergence is quite a strong condition which may be difficult to establish in practical situations. By contrast, by using the theory of distributions, we shall see that it is **always** possible to differentiate distributions and to interchange the limiting process with derivatives of any order.

1.2.2 Test functions, functionals and distributions

In equation (1.4) we found it convenient to think about the delta function in terms of its action on another function ϕ which is called a **test function**. The delta function may be regarded as a machine which operates on the test function ϕ to produce the number $\phi(0)$ as the result. We shall consider such machines (Figure 1.3) which take test functions as inputs and produce numbers as outputs in more detail.

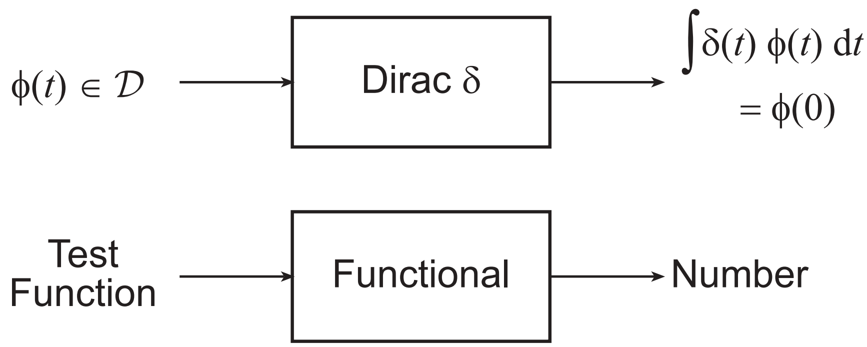

Figure 1.3 A Dirac delta function acts on a test function to return its value at zero. More generally, a functional is a machine which acts on a test function to produce a number.

We must first define the class of test functions on which our machines work. This class is denoted by $\mathcal{D}$, which we shall require to be a vector space. For technical reasons, we shall initially assume that test functions are taken from the class of **infinitely differentiable** functions which **vanish outside a finite interval**. It is important first to check that test functions exist.

Exercise: Show that if
$$h(t) = \begin{cases} \exp(-1/t) & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{1.18}$$
Then $\phi(t) = h(t)\,h(1-t)$ is a test function. (i.e., it is infinitely differentiable and vanishes outside a finite interval). In Figure 1.4, graphs of $h(t)$ and $\phi(t)$ are shown.

A **functional** T is a mapping from test functions to numbers. Given a test function $\phi \in \mathcal{D}$, we denote the number obtained by applying T to ϕ by the notation $\langle T, \phi\rangle$. We shall sometimes write this as $\langle T(t), \phi(t)\rangle$ to indicate explicitly the independent variable although T is **not** to be thought of as a **function** of t. Similarly we shall often refer to "the functional $T(t)$" by which we mean

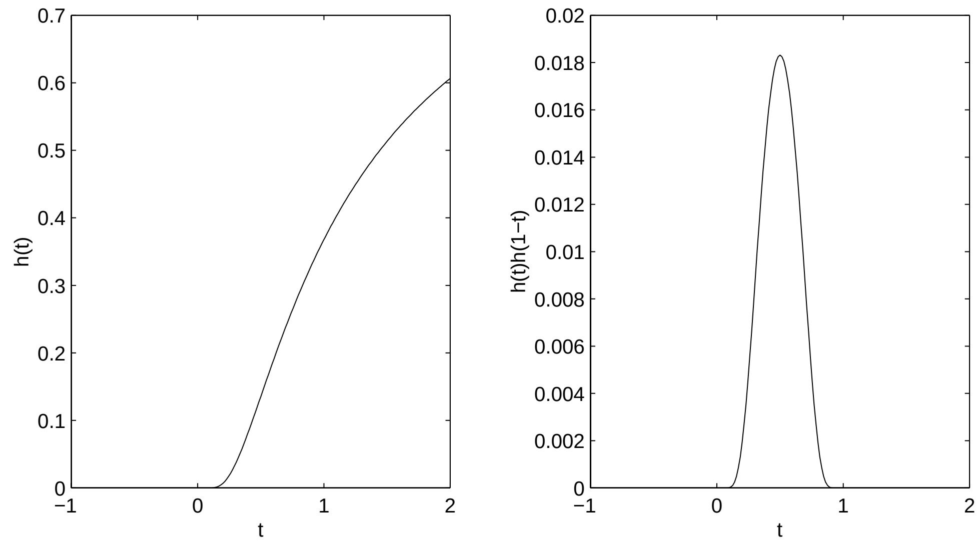

Figure 1.4 The function $h(t)$ is infinitely differentiable everywhere, and in particular at $t = 0$, although it is constant for $t < 0$. The function $h(t)\,h(1-t)$ is infinitely differentiable everywhere and vanishes identically outside the interval $[0, 1]$ and is an example of a test function.

that T acts on a test function which is a function of t. In particular, it is **not** meaningful to speak of "the value of T at t".

Two functionals S and T are said to be **equal** (with respect to the class of test functions $\mathcal{D}$) if for **every** test function $\phi \in \mathcal{D}$,

$$\langle S, \phi \rangle = \langle T, \phi \rangle. \tag{1.19}$$

A **distribution** is a functional which is both **linear** and **continuous**. A functional T is said to be **linear** if for any $\phi_1, \phi_2 \in \mathcal{D}$ and for any numbers c_1 and c_2, we have that

$$\langle T, c_1\phi_1 + c_2\phi_2 \rangle = c_1\langle T, \phi_1 \rangle + c_2\langle T, \phi_2 \rangle. \tag{1.20}$$

By analogy with the definition for an ordinary function to be continuous, the functional T is said to be **continuous** if for any sequence of test functions $\{\phi_n\}$ which converges to ϕ in $\mathcal{D}$,

$$\lim_{n\to\infty} \langle T, \phi_n \rangle = \langle T, \phi \rangle. \tag{1.21}$$

In order for this definition to make sense, we need to know what is meant by "a sequence of test functions converging in $\mathcal{D}$". It turns out to be convenient to define:

A sequence of test functions $\{\phi_n(t)\}$ in $\mathcal{D}$ **converges to zero** and we write $\phi_n \to 0$ if

1. for every $k \in \mathbb{N} \cup \{0\}$, the sequence of k'th derivatives $\phi_1^{(k)}(t), \phi_2^{(k)}(t), \ldots$ converges **uniformly** to zero.

2. there is a **common** interval $[a, b]$ independent of n such that every $\phi_n(t)$ vanishes outside of $[a, b]$.

We then say that $\{\phi_n\}$ **converges to** ϕ in $\mathcal{D}$ iff the sequence $\{\phi_n - \phi\}$ converges to zero in $\mathcal{D}$.

Note: For a **linear** functional T, the condition for continuity of T may be written in terms of sequences of test functions $\{\phi_n(t)\}$ which converge to zero in $\mathcal{D}$. The linear functional T is continuous iff for any such sequence of test functions $\phi_n \to 0$,

$$\lim_{n\to\infty} \langle T, \phi_n \rangle = 0. \tag{1.22}$$

Example: The **delta distribution** δ is defined by

$$\langle \delta, \phi \rangle = \phi(0). \tag{1.23}$$

Check that this is indeed a distribution by showing that it is both linear and continuous.

Exercise: With the definition of $h(t)$ given by equation (1.18), determine which of the following sequences of test functions converge to zero in $\mathcal{D}$ as $n \to \infty$:

1. $\alpha_n(t) = \frac{1}{n} h(t)\, h(1-t)$
2. $\beta_n(t) = \frac{1}{n} h(nt)\, h(1-nt)$
3. $\gamma_n(t) = \frac{1}{n} h(n-t)\, h(n+1-t)$.

1.2.3 Regular distributions

Suppose that $f(t)$ is a locally integrable function (i.e., the integral of $|f(t)|$ exists and is finite for every **finite** interval $[a, b]$). Then associated with the function f we may define a functional T_f via the rule

$$\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)\, \mathrm{d}t. \tag{1.24}$$

Since $\phi(t)$ is continuous on a closed and bounded interval, it is bounded, so the integral exists and is finite for any test function $\phi \in \mathcal{D}$. We now wish to prove that the functional is in fact a distribution.

Linearity follows from the linearity of integration. **Continuity** holds for if $\phi_n(t) \to 0$ in $\mathcal{D}$, and all the ϕ_n vanish outside $[a, b]$,

$$\left| \int_{-\infty}^{\infty} f(t)\phi_n(t)\, \mathrm{d}t \right| \le \int_a^b |f(t)||\phi_n(t)|\, \mathrm{d}t \le \max_{t\in\mathbb{R}} |\phi_n(t)| \int_a^b |f(t)|\, \mathrm{d}t.$$

As $n \to \infty$, the uniform convergence of $\phi_n(t)$ to the zero function means that the maximum of $|\phi_n(t)|$ converges to zero. Since f is locally integrable, the integral in the last term is finite, and so the entire right-hand side converges to zero.

Distributions which are induced by some locally integrable function are said to be **regular**. Other distributions (such as the delta distribution) are said to be **singular**. (As an exercise, prove that the delta distribution is not induced by any locally integrable function).

As mentioned above, two distributions S and T are said to be **equal** iff $\langle S, \phi \rangle = \langle T, \phi \rangle$ for all $\phi \in \mathcal{D}$. Notice that it is possible for two **different** locally integrable functions f and g to induce the **same** distribution, i.e., for

$$\int_{-\infty}^{\infty} f(t)\phi(t)\, \mathrm{d}t = \int_{-\infty}^{\infty} g(t)\phi(t)\, \mathrm{d}t \tag{1.25}$$

for all $\phi \in \mathcal{D}$. This will happen, for example, if $f(t)$ and $g(t)$ differ only on a set of measure zero, such as on a finite or countable set of points.

1.2.4 Some notational matters, generalized functions

Every locally integrable function $f(t)$ induces a regular distribution T_f. We deliberately wish to blur the distinction between the function $f(t)$ and the distribution it induces and so we shall often talk about "the distribution $f(t)$" to mean T_f and write $\langle f(t), \phi(t) \rangle$ instead of $\langle T_f, \phi \rangle$. Thus for locally integrable functions we shall write

$$\langle f(t), \phi(t) \rangle \equiv \langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)\,\mathrm{d}t. \tag{1.26}$$

Since two different locally integrable functions $f(t)$ and $g(t)$ can induce the same distribution if they differ on a set of measure zero, we shall say that "$f(t) = g(t)$ **distributionally**" to mean that they induce the same distribution, i.e., that for all $\phi \in \mathcal{D}$,

$$\langle f(t), \phi(t) \rangle = \langle g(t), \phi(t) \rangle. \tag{1.27}$$

This is to be understood as being quite distinct from the usual concept of equality of **functions**. Normally, we would write $f(t) = g(t)$ only if f and g agree at every point t in their common domain.

For singular distributions, such as the delta distribution δ there is **no** locally integrable function $\delta(t)$ satisfying

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,\mathrm{d}t = \langle \delta, \phi \rangle = \phi(0). \tag{1.28}$$

Nevertheless, we still use the **notation** above with the understanding that the integral form is simply an alternative way of writing the distribution. Unlike an ordinary function, we cannot evaluate $\delta(t)$ at an arbitrary value of t.

We shall always feel free to use the integral notation to denote the action of a distribution on a test function, whether the distribution is regular or singluar. We call the entity in the integrand which multiplies the test function a "generalized function".

1.3 Operations on distributions

In this section, we shall begin to describe how all this mathematical structure works. We shall use the regular distributions to tell us how to define **operations** on general distributions. These definitions will be chosen so as to be **consistent** with the usual operations on ordinary functions. The role of the test functions and the reasons for choosing them from such a restricted class (i.e., the infinitely differentiable functions which vanish outside a bounded set) will also become clearer.

1.3.1 Differentiation

Suppose that $f(t)$ is a function which is both locally integrable and continuously differentiable. It induces a distribution $\langle f, \phi \rangle$. The derivative $f'(t)$ of the function $f(t)$ is also locally integrable and induces another distribution $\langle f', \phi \rangle$. For consistency, we want the **distribution** f' to be the derivative of the **distribution** f whenever the **function** f' is the derivative of the **function** f.

For any test function $\phi \in \mathcal{D}$,

$$\langle f'(t), \phi(t) \rangle = \int_{-\infty}^{\infty} f'(t)\phi(t)\,\mathrm{d}t \tag{1.29}$$

$$= [f(t)\,\phi(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\phi'(t)\,\mathrm{d}t$$

$$= -\langle f(t), \phi'(t) \rangle, \tag{1.30}$$

where the boundary terms from the integration by parts have been set to zero since test functions vanish outside a bounded set. The fact that ϕ is infinitely differentiable means that ϕ' is also infinitely differentiable and is also in $\mathcal{D}$.

The above shows how we must define the derivative of a regular distribution. The notation suggests that we should extend this to define the derivative of an **arbitrary** distribution $T(t)$ to be the functional T' where

$$\langle T'(t), \phi(t) \rangle \equiv -\langle T(t), \phi'(t) \rangle. \tag{1.31}$$

We need to check that T' is in fact linear and continuous so that it is actually a distribution. Both of these facts follow readily from our definitions of $\mathcal{D}$ and convergence in this space.

Examples:

1. What is $u'(t)$? (**Note:** $u(t)$ is the unit step function defined by

$$u(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}. \tag{1.32}$$

The value of $u(0)$ is usually unspecified and can take on any value).

The function $u(t)$ is locally integrable and induces the regular distribution

$$\langle u(t), \phi(t) \rangle = \int_{-\infty}^{\infty} u(t)\phi(t)\,\mathrm{d}t = \int_{0}^{\infty} \phi(t)\,\mathrm{d}t. \tag{1.33}$$

The derivative of u is found by applying the definition

$$\langle u'(t), \phi(t) \rangle = -\langle u(t), \phi'(t) \rangle$$

$$= -\int_{0}^{\infty} \phi'(t)\,\mathrm{d}t = -\phi(\infty) + \phi(0)$$

$$= \phi(0) = \langle \delta(t), \phi(t) \rangle. \tag{1.34}$$

Since this holds for **all** $\phi \in \mathcal{D}$, we see that $u'(t) = \delta(t)$ in a distributional sense. This holds despite the fact that $u(t)$ is not differentiable in the ordinary sense at the point $t = 0$.

2. What is the derivative of $\delta(t)$? Again we simply use the definition. Given any test function $\phi \in \mathcal{D}$, the action of δ' on ϕ is given by

$$\langle \delta'(t), \phi(t) \rangle = -\langle \delta(t), \phi'(t) \rangle = -\phi'(0). \tag{1.35}$$

Formally, we write

$$\int_{-\infty}^{\infty} \delta'(t)\phi\,\mathrm{d}t = -\phi'(0). \tag{1.36}$$

This is a distribution called the **dipole** or **doublet**. Just as the charge density for a point charge can be written as a delta function, the charge density for an electric dipole can be written as a dipole function.

Similarly it is easy to show that

$$\langle \delta^{(k)}(t), \phi(t) \rangle = (-1)^k \langle \delta(t), \phi^{(k)}(t) \rangle = (-1)^k \phi^{(k)}(0). \tag{1.37}$$

These examples and the definition of the derivative of distributions show that every generalized function can be differentiated an infinite number of times. Thus we have not only extended our class of locally integrable functions (which can be discontinuous) to the generalized functions but in doing so have also made them all infinitely differentiable (in the distributional sense).

1.3.2 Shifting

Suppose that $T(t)$ is a distribution. How should we define the shifted distribution $T(t-a)$? Again we proceed via **consistency** with the regular distributions. Suppose that $f(t)$ is a locally integrable function. We want to see how the distribution induced by $f(t-a)$ is related to that induced by $f(t)$. Using the definition of a regular distribution in terms of the integral,

$$\langle f(t-a), \phi(t)\rangle = \int_{-\infty}^{\infty} f(t-a)\phi(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(u)\phi(u+a)\,\mathrm{d}u = \langle f(t), \phi(t+a)\rangle. \tag{1.38}$$

We thus make the definition:

Given any distribution $T(t)$, the functional $T(t-a)$ is defined by

$$\langle T(t-a), \phi(t)\rangle = \langle T(t), \phi(t+a)\rangle, \tag{1.39}$$

for any $\phi \in \mathcal{D}$. It is easy to check that the shifted distribution is also a distribution.

1.3.3 Scaling

Suppose that $T(t)$ is a distribution. We wish to assign a meaning to $T(at)$ for a real number a. Start once again with a locally integrable function $f(t)$. The distribution induced by $f(at)$ is

$$\int_{-\infty}^{\infty} f(at)\phi(t)\,\mathrm{d}t = \frac{1}{a}\int_{-a\infty}^{a\infty} f(u)\phi\left(\frac{u}{a}\right)\,\mathrm{d}u. \tag{1.40}$$

The limits on the right-hand side depend on the **sign** of a. If $a > 0$ this is

$$\frac{1}{a}\int_{-\infty}^{\infty} f(u)\phi\left(\frac{u}{a}\right)\,\mathrm{d}u = \frac{1}{a}\left\langle f(t), \phi\left(\frac{t}{a}\right)\right\rangle. \tag{1.41}$$

On the other hand if $a < 0$, the lower limit is ∞ and the upper limit is $-\infty$. To swap these limits around, we need to introduce an additional minus sign and so the right-hand side of (1.40) becomes

$$\frac{1}{a}\int_{\infty}^{-\infty} f(u)\phi\left(\frac{u}{a}\right)\,\mathrm{d}u = -\frac{1}{a}\left\langle f(t), \phi\left(\frac{t}{a}\right)\right\rangle. \tag{1.42}$$

These results may be summarized by the rule

$$\langle f(at), \phi(t)\rangle = \frac{1}{|a|}\left\langle f(t), \phi\left(\frac{t}{a}\right)\right\rangle. \tag{1.43}$$

This holds for real, nonzero values of a. We use this to define the scaling law for an arbitrary distribution $T(t)$

$$\langle T(at), \phi(t)\rangle = \frac{1}{|a|}\left\langle T(t), \phi\left(\frac{t}{a}\right)\right\rangle. \tag{1.44}$$

Examples:

1. The delta function is even:

$$\langle \delta(-t), \phi(t) \rangle = \frac{1}{|-1|} \left\langle \delta(t), \phi\left(\frac{t}{a}\right) \right\rangle = \phi(0) = \langle \delta(t), \phi(t) \rangle \tag{1.45}$$

2. The shifted delta function:

$$\langle \delta(t-\tau), \phi(\tau) \rangle = \langle \delta(\tau - t), \phi(\tau) \rangle = \langle \delta(\tau), \phi(\tau + t) \rangle = \phi(t) \tag{1.46}$$

Thus we may write

$$\int_{-\infty}^{\infty} \phi(\tau)\delta(t-\tau)\,\mathrm{d}\tau = \phi(t) \tag{1.47}$$

This equation holds at each point t for any test function $\phi \in \mathcal{D}$.

1.3.4 Product of a C^∞ function and a distribution

If $T(t)$ is a distribution and $g(t)$ is infinitely differentiable, $g(t)T(t)$ is defined by

$$\langle g(t)T(t), \phi(t) \rangle = \langle T(t), g(t)\phi(t) \rangle \tag{1.48}$$

It is easy to show that this is a distribution and that it is consistent with the result for regular distributions. We need $g(t)$ to be C^∞ to ensure that $g(t)\phi(t) \in \mathcal{D}$ whenever $\phi(t) \in \mathcal{D}$.

In general, it is not possible to define the product of two arbitrary distributions.

1.3.5 Integral of a delta function

Consider

$$f(t) = \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau \tag{1.49}$$

This simply means that we require that $f'(t) = \delta(t)$ and that $f(-\infty) = 0$. Both of these are satisfied by the unit step $u(t)$ which is also called the Heaviside function (sometimes written as $H(t)$).

Since $u(t)$ induces a regular distribution (as it is locally integrable), the successive integrals of $u(t)$ are ordinary functions such as the ramp $r(t) = t\,u(t)$, the parabola $p(t) = \frac{1}{2}t^2\,u(t)$ etc. These all induce regular distributions.

1.4 Sequences and series of distributions

In the previous section, we have seen how various mathematical operations defined on ordinary functions may be extended to apply to distributions. In physics, it is useful to develop an intuitive, pictorial understanding as well as a mathematical understanding of the theory. In this section we wish to see in what senses the intuitive picture of a delta function as an infinitely high spike of zero width but with unit area is a useful representation. For example, although there is no function which is infinitely tall and of zero width, we feel intuitively that a sequence of functions $f_n(t)$, such as

$$f_n(t) = \begin{cases} n & |t| < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}, \tag{1.50}$$

in some sense "tends to" a delta function, even though the sequence does not have a pointwise limit. As we shall see, the key idea is not to focus on the above as a sequence of functions, but rather on the sequence of regular distributions that the functions induce. To this end, we need to consider the properties of sequences of distributions and their convergence.

Suppose that $\{T_n(t)\}$ is a sequence of distributions and $\{T(t)\}$ is a distribution. We say that $T_n(t)$ **converges** to $T(t)$ and write $T_n(t) \to T(t)$ **distributionally** if

$$\lim_{n\to\infty} \langle T_n(t), \phi(t)\rangle = \langle T(t), \phi(t)\rangle, \tag{1.51}$$

for **all** test functions $\phi \in \mathcal{D}$.

Note: It is in fact unnecessary to assume that the limit T is a distribution. If for all $\phi \in \mathcal{D}$, $\langle T_n, \phi\rangle$ is a convergent series and we define the action of T in terms of this limit, T can be shown (with difficulty!) to be a distribution.

Theorem: If $\{S_n\}$ and $\{T_n\}$ are sequences of distributions which converge to S and T respectively, then

1. $aS_n + bT_n \to aS + bT$ for any constants a and b,
2. $T_n' \to T'$,
3. $T_n(at - b) \to T(at - b)$ for any real a and b,
4. $g(t)T_n(t) \to g(t)T(t)$ for any C^∞ function $g(t)$.

Perhaps the most surprising aspect of these relationships is the ease of the proofs. Consider for example the proof of the derivative result. For any test function $\phi \in \mathcal{D}$,

$$\langle T_n', \phi\rangle = -\langle T_n, \phi'\rangle \to -\langle T, \phi'\rangle = \langle T', \phi\rangle \tag{1.52}$$

These results are nonetheless remarkable, since they tell us that all the common distributional operations commute with the limiting process.

Let us again consider the sequence of functions (1.50), which we saw does not converge pointwise. For each n, $f_n(t)$ is locally integrable and so induces the regular distribution

$$\langle f_n(t), \phi(t)\rangle = n \int_{-1/(2n)}^{1/2n} \phi(t)\, \mathrm{d}t. \tag{1.53}$$

We wish to consider the **sequence** of distributions $\{f_n\}$. For any test function $\phi \in \mathcal{D}$, as $n \to \infty$, the sequence $\langle f_n(t), \phi(t)\rangle$ tends to $\phi(0)$ since ϕ is continuous at $t = 0$. By definition of the convergence of a sequence of distributions, we see that $f_n \to \delta$ distributionally which confirms our original intuition. This is an example of a sequence of functions which does not converge pointwise, but which does converge distributionally.

Let us next consider the sequence of functions

$$g_n(t) = \begin{cases} n^2 t & \text{for } 0 \le t \le \frac{1}{n} \\ n^2\left(\frac{2}{n} - t\right) & \text{for } \frac{1}{n} < t \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases} \tag{1.54}$$

which, as we discovered (see Figure 1.2), converges pointwise to zero. Given a test function $\phi \in \mathcal{D}$, we see that

$$\int_{-\infty}^{\infty} g_n(t)\phi(t)\ \mathrm{d}t = n^2 \int_0^{1/n} t\phi(t)\ \mathrm{d}t + n^2 \int_{1/n}^{2/n} \left(\frac{2}{n} - t\right) \phi(t)\ \mathrm{d}t \tag{1.55}$$

We now wish to show that each of the terms on the right hand side is equal to $\frac{1}{2}\phi(0)$. Since

$$n^2 \int_0^{1/n} t \, \mathrm{d}t = \frac{1}{2}, \tag{1.56}$$

we see that

$$\begin{aligned} \left| n^2 \int_0^{1/n} t\phi(t) \, \mathrm{d}t - \frac{1}{2}\phi(0) \right| &= \left| n^2 \int_0^{1/n} t\left(\phi(t) - \phi(0)\right) \mathrm{d}t \right| \\ &\leq n^2 \int_0^{1/n} |t| \left|\phi(t) - \phi(0)\right| \mathrm{d}t. \end{aligned} \tag{1.57}$$

Since ϕ is a test function, it is continuous at $t = 0$. By definition of continuity, given any $\varepsilon > 0$, we can find $\delta > 0$ such that whenever $|t| < \delta$, $|\phi(t) - \phi(0)| < \varepsilon$. So if we choose any $n > \delta^{-1}$,

$$\left| n^2 \int_0^{1/n} t\phi(t) \, \mathrm{d}t - \frac{1}{2}\phi(0) \right| \leq n^2 \int_0^{1/n} |t| \, \varepsilon \mathrm{d}t = \frac{\varepsilon}{2}. \tag{1.58}$$

This establishes that

$$\lim_{n\to\infty} n^2 \int_0^{1/n} t\phi(t) \, \mathrm{d}t = \frac{1}{2}\phi(0). \tag{1.59}$$

Similarly, you should check that you can show

$$\lim_{n\to\infty} n^2 \int_{1/n}^{2/n} \left(\frac{2}{n} - t\right) \phi(t) \, \mathrm{d}t = \frac{1}{2}\phi(0), \tag{1.60}$$

so that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(t)\phi(t) \, \mathrm{d}t = \phi(0). \tag{1.61}$$

This establishes that the sequence of distributions $\{g_n\}$ induced by $g_n(t)$ also tends distributionally to the Dirac δ.distribution. Notice that the distributional limit of a sequence can be very different from the pointwise limit. It should also be clear that there are an infinite number of sequences of regular distributions which converge distributionally to the delta distribution. When contemplating the Dirac delta function, one should try to think of it as the common limit of all such sequences of distributions.

Exercise: Show that the sequence of functions (see figure 1.5)

$$h_n(t) = \frac{1}{\pi}\left(\frac{n}{n^2t^2 + 1}\right) \tag{1.62}$$

converges distributionally to $\delta(t)$ as $n \to \infty$.

Let us now see how the theorem helps us to visualize what we mean by the derivative of a delta function, which we denoted by $\delta'(t)$. According to the theorem if we can find **any** sequence of distributions $\{T_n\}$ with the property that $T_n \to \delta$ distributionally, we are guaranteed that T_n' will also converge to the distribution δ'. We have found three sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$, all of which tend distributionally to δ. By differentiating each of these, we will get three sequences of distributions which tend to $\delta'(t)$. It is straightforward to find $h_n'(t)$ and $g_n'(t)$, which are

$$h_n'(t) = -\frac{2}{\pi}\frac{n^3 t}{\left(n^2t^2 + 1\right)^2} \tag{1.63}$$

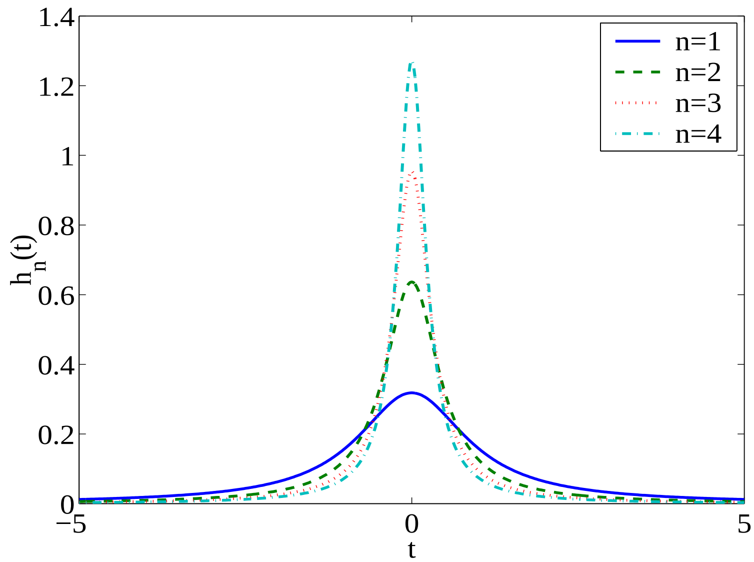

Figure 1.5 A sequence $h_n(t)$ which tends distributionally to $\delta(t)$.

and

$$g_n'(t) = \begin{cases} n^2 & \text{for } 0 \le t \le \frac{1}{n} \\ -n^2 & \text{for } \frac{1}{n} < t \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

A graph of $h_n'(t)$ for $n = 1, ..., 4$ is shown in Figure 1.6.

The common feature of g_n' and h_n' is that as $n \to \infty$, they both have a large positive peak followed by a large negative peak. The areas of each of the two peaks increases proportionally to n. We can also find the sequence of distributions $f_n'(t)$, each of which turns out to be singular, since the functions f_n are not differentiable. We may write $f_n(t)$ as the difference of two unit step functions,

$$\begin{aligned} f_n(t) &= \begin{cases} n & |t| < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases} \\ &= n\left[u\left(t + \frac{1}{2n}\right) - u\left(t - \frac{1}{2n}\right)\right] \end{aligned} \tag{1.64}$$

From our previous result (1.34) that $u'(t) = \delta(t)$ distributionally, we see that

$$f_n'(t) = n\left[\delta\left(t + \frac{1}{2n}\right) - \delta\left(t - \frac{1}{2n}\right)\right], \tag{1.65}$$

which consists of two delta functions of strengths $\pm n$ located at distance $1/n$ apart. As $n \to \infty$, the two spikes get closer to each other and their areas become larger. From this we can see why δ' is referred to as a dipole.

Given a test function $\phi(t) \in \mathcal{D}$, it is clear that

$$\begin{aligned} \left\langle f_n'(t), \phi(t) \right\rangle &= \left\langle n\left[\delta\left(t + \frac{1}{2n}\right) - \delta\left(t - \frac{1}{2n}\right)\right], \phi(t) \right\rangle \\ &= n\left\langle \delta\left(t + \frac{1}{2n}\right), \phi(t) \right\rangle - n\left\langle \delta\left(t - \frac{1}{2n}\right), \phi(t) \right\rangle \\ &= n\left[\phi\left(-\frac{1}{2n}\right) - \phi\left(\frac{1}{2n}\right)\right]. \end{aligned} \tag{1.66}$$

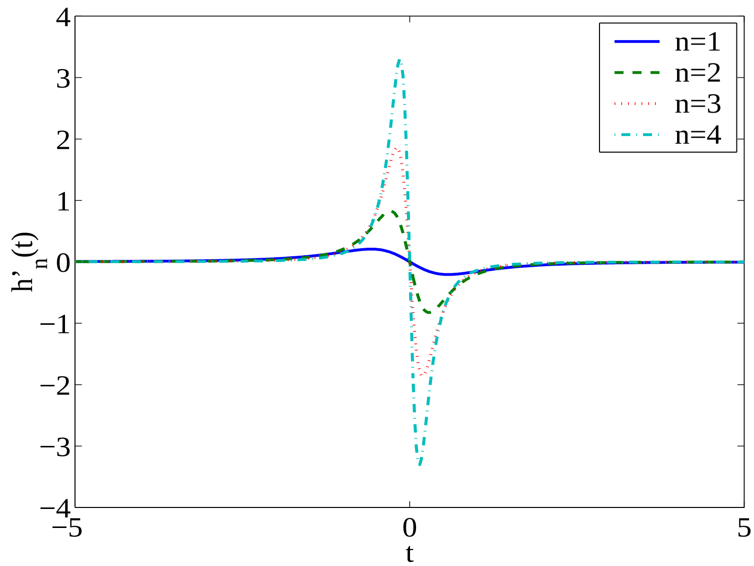

Figure 1.6 A sequence $h_n'(t)$ which tends distributionally to $\delta'(t)$, the derivative of the Dirac δ function.

Taking the limit as $n \to \infty$, we see that

$$\lim_{n\to\infty} \left\langle f_n'(t), \phi(t) \right\rangle = -\phi'(0), \tag{1.67}$$

which is consistent with what we found (1.35) from the mathematical definition of the derivative of a distribution.

These results often allow us to visualize arbitrary (singular) distributions since we can always express such a distribution as the limit of a sequence of regular distributions associated with a sequence of locally integrable functions. It is possible to calculate with the singular distribution by taking limits over the results obtained with the sequence of regular distributions.

Exercise: Show that for any test function $\phi(t)$,

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \phi(t) \cos(n^2 t)\, \mathrm{d}t = 0 \tag{1.68}$$

We can similarly treat a **series** of distributions $\sum_{k=1}^{\infty} T_k$ which is said to converge to the distribution S if the sequence of numbers $\langle \sum_{k=1}^{n} T_k(t), \phi(t) \rangle$ converges as $n \to \infty$ for every $\phi \in \mathcal{D}$. We may treat a series in terms of the sequence of its partial sums and apply the above results for sequences. In particular we can differentiate a series of distributions term-by-term, i.e.,

$$\left\langle \left(\sum_{k=1}^{\infty} T_k \right)', \phi \right\rangle = \sum_{k=1}^{\infty} \langle T_k', \phi \rangle \tag{1.69}$$

Using a Riemann approach to integration as a limit of a summation, we may use linearity to write distributional equations such as

$$f(t) = \int_{-\infty}^{\infty} f(\tau) \delta(t - \tau)\, \mathrm{d}\tau \tag{1.70}$$

where f is now regarded as a locally integrable function. What this means is that for all $\phi \in \mathcal{D}$,

$$\left\langle \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau, \phi(t) \right\rangle = \langle f(t), \phi(t) \rangle \tag{1.71}$$

Notice the difference between this distributional equality and the previous pointwise relationship (1.47).

We may regard (1.70) as expressing $f(t)$ as a linear combination of impulses. It may be helpful to think of the analogy of writing a vector as a linear combination of basis vectors, e.g.,

$$\mathbf{v} = \sum v_i \mathbf{e}_i$$

The delta functions are the basis vectors, indexed by their location τ. The sum is over the index of these basis vectors τ which correspond to i. The numbers $f(\tau)\,\mathrm{d}\tau$ are analogous to the coefficients of the expansion v_i, and the resulting function f is analogous to the vector $\mathbf{v}$.

1.5 Differential equations involving distributions

We wish to consider differential equations in which the solutions and driving terms are generalized functions. We shall mainly be concerned with linear differential equations with homogeneous boundary and/or initial conditions.

As a specific example, consider the problem of a string of length L under tension T constrained so that its transverse displacement $u(x)$ is zero at the endpoints. i.e., $u(0) = u(l) = 0$. A distribution of transverse "pressure" $p(x)$ (actually a force per unit length) is applied to the string and we wish to determine the resulting displacement. The differential equation is

$$\mathcal{L}u = -T\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = p(x), \qquad u(0) = u(l) = 0 \tag{1.72}$$

where $\mathcal{L}$ denotes the differential operator

$$\mathcal{L} = -T\frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{1.73}$$

For each distribution of pressure $p(x)$ there is a resulting displacement $u(x)$. We can treat the equation like a physical system with $p(x)$ as an input function and $u(x)$ as the output function. We see that the system is **linear**, i.e., if $u_1(x)$ is the displacement for pressure $p_1(x)$ and $u_2(x)$ is the displacement for pressure $p_2(x)$, the displacement if we apply the pressure $c_1 p_1(x) + c_2 p_2(x)$ is $c_1 u_1(x) + c_2 u_2(x)$. This follows from the linearity of $\mathcal{L}$ and the homogeneity of the boundary conditions. Linearity allows us to apply the following important idea:

We know that for any locally integrable function $p(x)$, we can write $p(x)$ as a linear combination of impulses i.e.,

$$p(x) = \int_{-\infty}^{\infty} p(\xi)\delta(x-\xi)\,\mathrm{d}\xi \tag{1.74}$$

So if we know how the physical system responds to the input $\delta(x-\xi)$, we can use this to work out how it will respond to $p(x)$ which is simply a linear combination of these delta function inputs.

Let us suppose that we can find the solution $h(x|\xi)$ to the problem

$$\mathcal{L}h(x|\xi) = \delta(x-\xi), \qquad h(0|\xi) = h(l|\xi) = 0 \tag{1.75}$$

where the input to the system is a delta function. Then by linearity the output for input $p(x)$ must be given by

$$u(x) = \int_{-\infty}^{\infty} p(\xi) h(x|\xi)\, \mathrm{d}\xi \tag{1.76}$$

which is simply (1.74) with the delta function replaced by the response that it produces when it is "processed" by the system.

The function $h(x|\xi)$ is called the **impulse response** of the system. It is also known as the **Green's function** of the boundary value problem.

Example: Calculating the Green's function for

$$-T \frac{\mathrm{d}^2 h(x|\xi)}{\mathrm{d}x^2} = \delta(x-\xi), \qquad h(0|\xi) = h(l|\xi) = 0 \tag{1.77}$$

This corresponds to the application of a concentrated force at $x = \xi$. In this case $h(x|\xi)$ can be found by direct integration (figure 1.7). We see that

$$\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = -\frac{1}{T}\delta(x-\xi) \tag{1.78}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} = c_1 - \frac{1}{T} u(x-\xi) \tag{1.79}$$

$$h(x|\xi) = \begin{cases} c_2 + c_1 x & \text{for } x < \xi \\ c_2 + c_1 x - \frac{1}{T}(x-\xi) & \text{for } x > \xi \end{cases} \tag{1.80}$$

The quantities c_1 and c_2 are constants of integration. For all values of c_1 and c_2, the function $h\,(x|\xi)$ satisfies the differential equation, but the boundary conditions are only satisfied for a special choice of the constants. Applying the boundary conditions, we find that $c_2 = 0$ and $c_1 = (l-\xi)/(lT)$. Hence

$$h(x|\xi) = \begin{cases} \dfrac{x\,(l-\xi)}{lT} & \text{for } x < \xi \\ \dfrac{\xi\,(l-x)}{lT} & \text{for } x > \xi \end{cases} \tag{1.81}$$

Notice that

1. In regions $x < \xi$ and $x > \xi$, the Green's function satisfies the homogeneous equation

$$\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = 0 \tag{1.82}$$

2. At $x = \xi$, $h(x|\xi)$ is continuous and $\partial h/\partial x$ has a jump discontinuity to give the delta function in the second derivative

$$\frac{\mathrm{d}h}{\mathrm{d}x}(\xi^+|\xi) - \frac{\mathrm{d}h}{\mathrm{d}x}(\xi^-|\xi) = -\frac{1}{T} \tag{1.83}$$

3. The solution of the boundary value problem for an arbitrary $p(x)$ is given by

$$u(x) = \int_0^l p(\xi) h(x|\xi)\, \mathrm{d}\xi \tag{1.84}$$

$$= \int_0^x p(\xi) \frac{\xi\,(l-x)}{lT}\, \mathrm{d}\xi + \int_x^l p(\xi) \frac{x\,(l-\xi)}{lT}\, \mathrm{d}\xi \tag{1.85}$$

Exercise: Check that this is a solution of the boundary value problem. You may find the following result (which you should also prove if you have not seen it before) useful:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(t,x)\, \mathrm{d}t = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(t,x)\, \mathrm{d}t - f(a(x),x)\, a'(x) + f(b(x),x)\, b'(x) \tag{1.86}$$

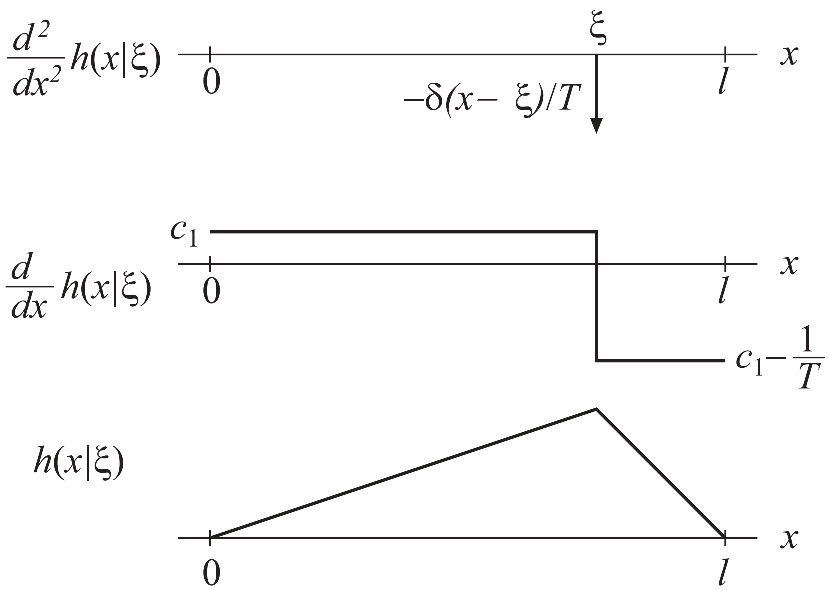

Figure 1.7 Calculation of Green's function by direct integration. Values of constants are found by satisfying the boundary conditions.

1.5.1 Green's function for regular n'th order linear differential equations with homogeneous initial/boundary conditions

The above example may be generalized to the following. Consider the linear n'th order differential equation on the interval I

$$\mathcal{L}g = a_n(x)\frac{\mathrm{d}^n g}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} g}{\mathrm{d}x^{n-1}} + ... + a_1(x)\frac{\mathrm{d}g}{\mathrm{d}x} + a_0(x)g = f(x) \tag{1.87}$$

together with n homogeneous boundary or initial conditions (which constrain u and/or its derivatives at the endpoints of I). We assume that $a_n(x) \neq 0$ for all $x \in I$ so that the equation is regular.

The Green's function $h(x|\xi)$ is the solution of

$$\mathcal{L}h(x|\xi) = \delta(x - \xi) \tag{1.88}$$

where

1. $h(x|\xi)$ satisfies the homogeneous equation (with zero right-hand side) for $x < \xi$ and for $x > \xi$,
2. $h(x|\xi)$ satisfies all the boundary or initial conditions for each ξ,
3. The functions h, $\partial h/\partial x$,..., $\partial^{n-2}h/\partial x^{n-2}$ are continuous throughout I and in particular at $x = \xi$.
4. $\partial^{n-1}h/\partial x^{n-1}$ has a jump discontinuity at $x = \xi$ and

$$\frac{\mathrm{d}^{n-1}h}{\mathrm{d}x^{n-1}}(\xi^+|\xi) - \frac{\mathrm{d}^{n-1}h}{\mathrm{d}x^{n-1}}(\xi^-|\xi) = \frac{1}{a_n(\xi)} \tag{1.89}$$

The solution to the original problem can be found for any $f(x)$ and is

$$g(x) = \int_I f(\xi) h(x|\xi)\, \mathrm{d}\xi \tag{1.90}$$

Example: Calculate the Green's function for the boundary value problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + k^2 u = f(x), \qquad u(0) = u(l) = 0 \tag{1.91}$$

and use the Green's function to find the solution for $f(x) = x$.

The Green's function $h(x|\xi)$ is a solution of

$$\frac{\mathrm{d}^2 h(x|\xi)}{\mathrm{d}x^2} + k^2 h(x|\xi) = \delta(x - \xi), \qquad h(0|\xi) = h(l|\xi) = 0 \tag{1.92}$$

Where the "spike" of the delta function is located at $x = \xi$. In order to find h, we consider the two regions $x < \xi$ and $x > \xi$ separately. In each of the two regions, h is a solution of the homogeneous differential equation, but with different constants. Hence

$$h(x|\xi) = \begin{cases} A\cos kx + B\sin kx & \text{for } x < \xi \\ C\cos kx + D\sin kx & \text{for } x > \xi \end{cases} \tag{1.93}$$

We have to choose the constants A, B, C and D (which all depend on ξ) to satisfy the boundary conditions and the "joining conditions" at $x = \xi$. The boundary condition at zero tells us that

$$A = 0, \tag{1.94}$$

since $x = 0 < \xi$, and we have to use the upper expression for h. The boundary condition at $x = l > \xi$ imposes constraints on the lower expression for h. In particular

$$C\cos kl + D\sin kl = 0. \tag{1.95}$$

The joining conditions tell us firstly that $h(x|\xi)$ is continuous at $x = \xi$ since h is differentiable there. Approaching $x = \xi$ from below and above and equating the results shows us that

$$A\cos k\xi + B\sin k\xi = C\cos k\xi + D\sin k\xi. \tag{1.96}$$

The first derivative $\partial h/\partial x$ is however discontinuous at $x = \xi$ since the second derivative has a delta function at ξ. The size of the jump discontinuity is equal to the area under the delta function, so that $\partial h(\xi^+|\xi)/\partial x - \partial h(\xi^-|\xi)/\partial x = 1$. In terms of the expressions above

$$(-kC\sin k\xi + kD\cos k\xi) - (-kA\sin k\xi + kB\cos k\xi) = 1 \tag{1.97}$$

Solving equations (1.94) through (1.97) simultaneously yields

$$A = 0,\ B = -\frac{\sin k(l-\xi)}{k\sin kl},\ C = -\frac{\sin k\xi}{k} \text{ and } D = \frac{\sin k\xi \cos kl}{k\sin kl} \tag{1.98}$$

Substituting back into the expression for $h(x|\xi)$ and simplifying, we find

$$h(x|\xi) = \begin{cases} -\dfrac{\sin k(l-\xi)\sin kx}{k\sin kl} & \text{for } x < \xi \\ -\dfrac{\sin k\xi \sin k(l-x)}{k\sin kl} & \text{for } x > \xi \end{cases}. \tag{1.99}$$

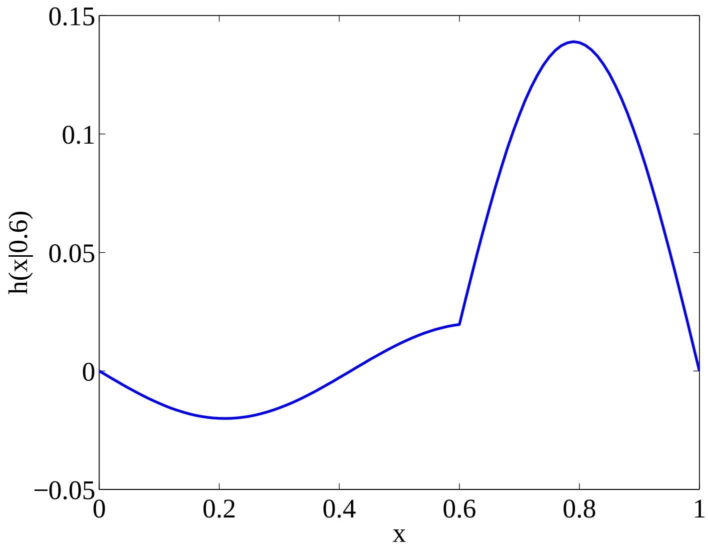

Figure 1.8 Green's function $h(x|\xi)$ for $\xi = 0.6$, $k = 7.5$ and $l = 1$. Notice that the function is continuous at $x = \xi$, but that the derivative has a jump of unit size there.

In this case, we note that the Green's function is symmetric, i.e., that $h(x|\xi) = h(\xi|x)$. This is the case if the boundary value problem is **self-adjoint**. Symmetric Green's functions can be written more compactly in terms of the variables $x_< = \min(x, \xi)$ and $x_> = \max(x, \xi)$. In the example,

$$h(x|\xi) = \frac{-\sin kx_< \sin k(l - x_>)}{k \sin kl} \tag{1.100}$$

This Green's function is plotted in Figure 1.8.

We now wish to solve the boundary value problem $\mathcal{L}u = f$ with $f(x) = x$. By linearity, we have that

$$u(x) = \int_0^l h(x|\xi) f(\xi)\, \mathrm{d}\xi = \int_0^l \xi h(x|\xi)\, \mathrm{d}\xi \tag{1.101}$$

Since $h(x|\xi)$ has different functional forms depending on whether $x < \xi$ or $x > \xi$, we split the range of integration into two. Notice carefully that the integration variable is ξ, so that the interval needs to be split into $[0, x]$ and $[x, l]$. Thus,

$$u(x) = \int_0^x \xi h(x|\xi)\, \mathrm{d}\xi + \int_x^l \xi h(x|\xi)\, \mathrm{d}\xi. \tag{1.102}$$

Consider the first integral. In the interval $\xi \in [0, x]$, we have that $\xi < x$ and so we must use the **lower** of the two expressions for $h(x|\xi)$. For the second integral, in the interval $\xi \in [x, l]$, we have that $\xi > x$ and so we must use the **upper** of the two expressions of $h(x|\xi)$. Substituting the appropriate formulae gives

$$\begin{aligned} u(x) &= -\int_0^x \xi \frac{\sin k\xi \sin k(l-x)}{k \sin kl}\, \mathrm{d}\xi - \int_x^l \xi \frac{\sin k(l-\xi) \sin kx}{k \sin kl}\, \mathrm{d}\xi \\ &= -\frac{\sin k(l-x)}{k \sin kl} \int_0^x \xi \sin k\xi\, \mathrm{d}\xi - \frac{\sin kx}{k \sin kl} \int_x^l \xi \sin k(l-\xi)\, \mathrm{d}\xi \\ &= \frac{x \sin kl - l \sin kx}{k^2 \sin kl}. \end{aligned}$$

It is easy to check that this satisfies the boundary conditions $u(0) = u(l) = 0$ and $u''(x) + k^2 u(x) = x$, as required.

Exercise: Calculate the Green's function for the following problems

1. $$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + k^2 u = f(x), \qquad u(0) = u'(l) = 0$$

   and use the Green's function to find the solution for $f(x) = 1$.

2. $$RC\frac{\mathrm{d}v_C}{\mathrm{d}t} + v_C = v_I, \qquad v_C(-\infty) = 0$$

   This is the impulse response of a RC quasi-integrator circuit. Calculate the response if $v_I(t) = \cos 2\pi\nu t$.

3. $$LC\frac{\mathrm{d}^2 v_C}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C}{\mathrm{d}t} + v_C = v_I, \qquad v_C(-\infty) = v_C'(-\infty) = 0$$

   This is the impulse response of a series LCR circuit when the output is taken across the capacitor. Calculate the response when the circuit is excited with a unit step input voltage.

References:

Distribution theory is covered in many texts with classification number 515.782. Two readable accounts are

Theory of Distributions, A non-technical introduction by J.I. Richards and H.K. Youn, (Cambridge University Press, 1990).

Distribution Theory and Transform Analysis by A.H. Zemanian, (McGraw-Hill, 1965).

A useful book for mathematical background is

Mathematical Analysis by T.M. Apostol, (Addison-Wesley, 1974).